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The magic of cross-spectrum measurements from DC to optics

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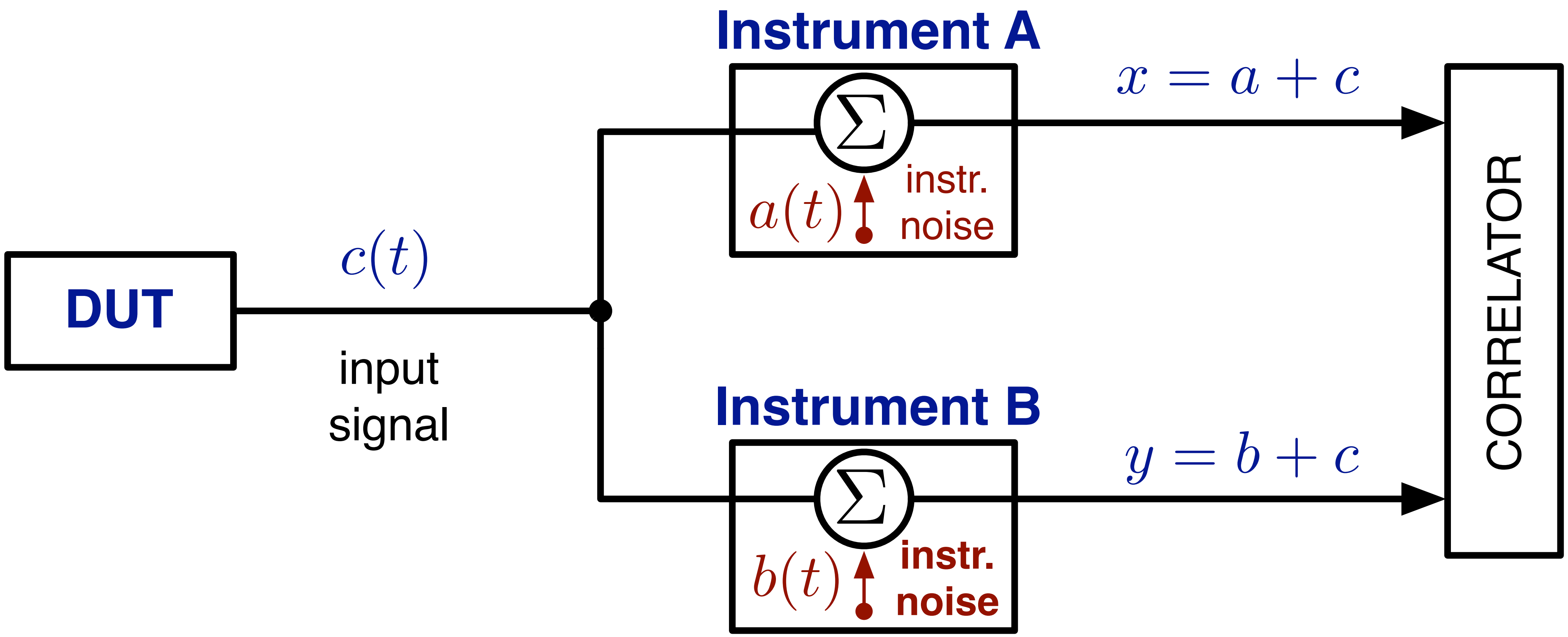
Outline

1. Theory
   - Basics
   - Rejection of the background noise
   - Examples
2. Applications
   - Radio-astronomy, radiometry, and thermometry
   - AM-PM noise
   - Other applications

home page http://rubiola.org

Part 1 – Theory

The main idea



- Two instruments measure independently the same physical quantity
- **Averaging must help to reject the instrument noise, and measure the statistical properties of the signal**

Notation: Fourier transform

x(t) <=> X(ıf) = X'(ıf)+ ıX"(ıf)

Ergodicity

FFT => sequence of discrete spectra

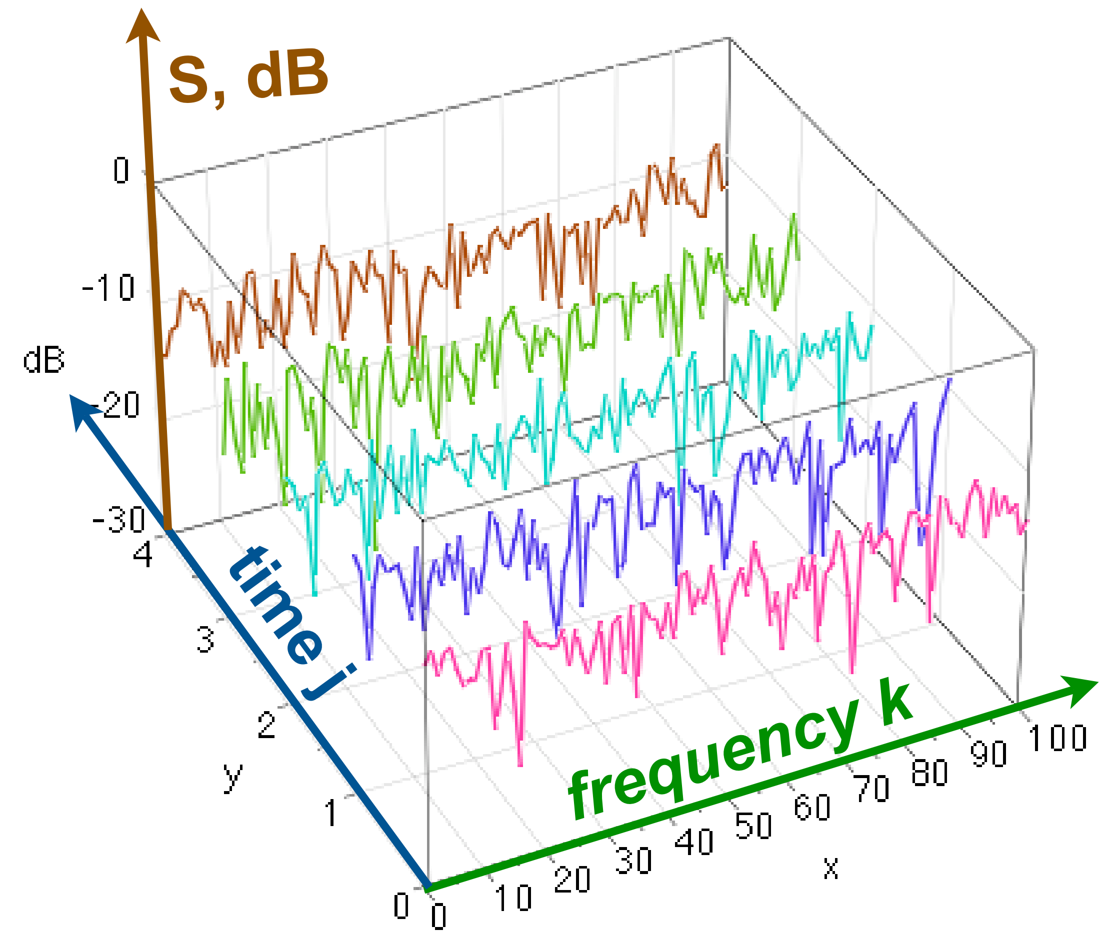
analog	sample no	spectra seq.
$S(f)$	$S_j(f)$	S_{jk}

integer time

integer frequency

lock j=J & run k: S_{Jk} is a spectrum

run j & lock k=K: S_{jK} is a time series



white noise: $S(f_1)$ and $S(f_2)$, $f_1 \neq f_2$, are uncorrelated, hence

given i, S_k can be seen as the ensemble (at a given time)

Ergodicity allows to interchange time-statistics with ensemble statistics.

Sweeping the frequency, we get the statistical behavior of the time series.
No need for forthcoming samples. Useful when S is a large-size average.

flicker noise: need $f_1 \neq\neq f_2$, for $S(f_1)$ and $S(f_2)$, to be uncorrelated (less deg. of freedom)

Single-channel spectrum Sxx

gaussian X with independent Re and Im

Spectrum

$$\langle S_{xx} \rangle_m = \langle X X^* \rangle_m$$
$$= \langle (X' + \imath X'') \times (X' - \imath X'') \rangle_m$$
$$= \langle (X')^2 + (X'')^2 \rangle_m$$

white, gaussian, avg = 0, var = 1/2

white, χ^2, with 2m degrees of freedom avg = 1, var = 1/m

$$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{1}{m}}$$

the S_{xx} track on the FFT-SA shrinks as $1/m^{1/2}$

Normalization: in 1 Hz bandwidth
var{X} = 1, and var{X'} = var{X"} = 1/2

Syx with correlated term C≠0 (1)

gaussian A, B, C with independent Re and Im

Cross-spectrum

$$\langle S_{yx}\rangle_m = \langle Y X^*\rangle_m = \langle (Y' + \imath Y'') \times (X' - \imath X'')\rangle_m = \langle [Y'X' + Y''X''] + \imath\,[Y''X' - Y'X'']\rangle_m$$

Expand

$$X = (A' + \imath A'') + (C' + \imath C'') \quad \text{and} \quad Y = (B' + \imath B'') + (C' + \imath C'')$$

Split $\langle S_{yx}\rangle_m = \langle S_{yx}\rangle_m\big|_{\text{instr}} + \langle S_{yx}\rangle_m\big|_{\text{mixed}} + \langle S_{yx}\rangle_m\big|_{\text{DUT}}$

#1 $\langle S_{yx}\rangle_m\big|_{\text{instr}} = \langle B'A' + B''A''\rangle_m + \imath\,\langle B''A' + B'A''\rangle_m$

#2 $\langle S_{yx}\rangle_m\big|_{\text{mixed}} = \langle B'C' + B''C'' + C'A' + C''A''\rangle_m + \imath\,\langle B''C' - B'C'' + C''A' - C'A''\rangle_m$

#3 $\langle S_{yx}\rangle_m\big|_{\text{DUT}} = \langle (C')^2 + (C'')^2\rangle_m$

The useful signal C is real, the noise terms are complex. Take Re{S_{yx}}
(Yet there can be some risk!)

Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
hence var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Syx with correlated term C≠0 (2)

gaussian A, B, C with independent Re and Im

#1 $\langle S_{yx}\rangle_m \big|_{\text{instr}} = \langle B'A' + B''A''\rangle_m + \imath\,\langle B''A' + B'A''\rangle_m$

white, gaussian, avg = 0, var = 1/4

white, gaussian, avg = 0, var = 1/2m

#2 $\langle S_{yx}\rangle_m \big|_{\text{mixed}} = \langle B'C' + B''C'' + C'A' + C''A''\rangle_m + \imath\,\langle B''C' - B'C'' + C''A' - C'A''\rangle_m$

white, gaussian, avg = 0, var = $\kappa^2/4$

white, gaussian, avg = 0, var = κ^2/m

white, gaussian, avg = 0, var = $1/2\kappa^2$

#3 $\langle S_{yx}\rangle_m \big|_{\text{DUT}} = \langle (C')^2 + (C'')^2\rangle_m$

white, χ^2, with 2m deg. of freedom avg = κ^2, var = κ^4/m

#3 $\dfrac{\text{dev}}{\text{avg}} = \sqrt{\dfrac{1}{m}}$ **at large m the noise terms vanish, and the S_{yx} track on the FFT-SA shrinks as $1/m^{1/2}$**

Normalization: in 1 Hz bandwidth var{A} = var{B} = 1, var{C}=κ^2
hence var{A'} = var{A"} = var{B'} = var{B"} = 1/2, and var{C'} = var{C"} = $\kappa^2/2$

Detection, and noise-rejection law

Gaussian X, Y, independent (C=0). Re and Im are independent

Real part

$$\Re\{\langle S_{yx}\rangle_m\} = \langle Y'X' + Y''X''\rangle_m$$

$Y'X'$, $Y''X''$: white, gaussian, avg = 0, var = 1/4

white, gaussian

+ unbiased
+ fastest convergence
– can't use log scale (dB!)

$$\text{avg} = 0 \qquad \text{var} = \frac{1}{2m}$$

Abs Real part

$$|\Re\{\langle S_{yx}\rangle_m\}| = |\langle Y'X' + Y''X''\rangle_m|$$

$Y'X'$, $Y''X''$: white, gaussian, avg = 0, var = 1/4

white, one-sided gaussian,

– biased
= good convergence
+ can use log scale (dB!)

$$\text{avg} = \sqrt{\frac{1}{\pi m}} \qquad \text{var} = \left(\frac{1}{2} - \frac{1}{\pi}\right)\frac{1}{m}$$

Modulus

$$|\langle S_{yx}\rangle|_m = \sqrt{\left[\langle Y'X'\rangle_m + \langle Y''X''\rangle_m\right]^2 + \left[\langle Y''X'\rangle_m - \langle Y'X''\rangle_m\right]^2}$$

$Y'X'$, $Y''X''$, $Y''X'$, $Y'X''$: white, gaussian, avg = 0, var = 1/4

$\langle Y'X'\rangle_m + \langle Y''X''\rangle_m$ and $\langle Y''X'\rangle_m - \langle Y'X''\rangle_m$: white, gaussian, avg = 0, var = 1/2m

white, Rayleigh

– biased
– slowest convergence
+ can use log scale (dB!)

$$\text{avg} = \sqrt{\frac{\pi}{4m}} \qquad \text{var} = \left(1 - \frac{\pi}{4}\right)\frac{1}{m}$$

Normalization: in 1 Hz bandwidth var{X} = var{Y} = 1, and var{X'} = var{X"} = var{Y'} = var{Y"} = 1/2

Noise rejection, |Syx| and |Re{Syx}|

Independent X and Y, var{X} = var{Y}= 1/2

|Syx| => Rayleigh distribution

average $$\mathbb{E}\{S\} = \sqrt{\frac{\pi}{4m}}$$ = 0.886/√m

$|\langle Syx \rangle_m| \sim -5 \log_{10}(m) - 0.53$ dB

deviation

$$\sqrt{\mathbb{E}\left\{|S - \mathbb{E}\{S\}|^2\right\}} = \sqrt{\left(1 - \frac{\pi}{4}\right)\frac{1}{m}}$$

= √(0.215/m)

the dev / avg ratio is independent of m

$$\frac{\sqrt{\mathbb{E}\{|S - \mathbb{E}\{S\}|^2\}}}{\mathbb{E}\{S\}} = \sqrt{\frac{4}{\pi} - 1}$$ = 0.523

|Re{Syx}| => one-sided gaussian distrib.

average $$\mathbb{E}\{S\} = \sqrt{\frac{1}{\pi m}}$$ = 0.564/√m

$|\langle Re\{Syx\rangle_m\}| \sim -5 \log_{10}(m) - 2.49$ dB

deviation

$$\sqrt{\mathbb{E}\left\{|S - \mathbb{E}\{S\}|^2\right\}} = \sqrt{\left(\frac{1}{2} - \frac{1}{\pi}\right)\frac{1}{m}}$$

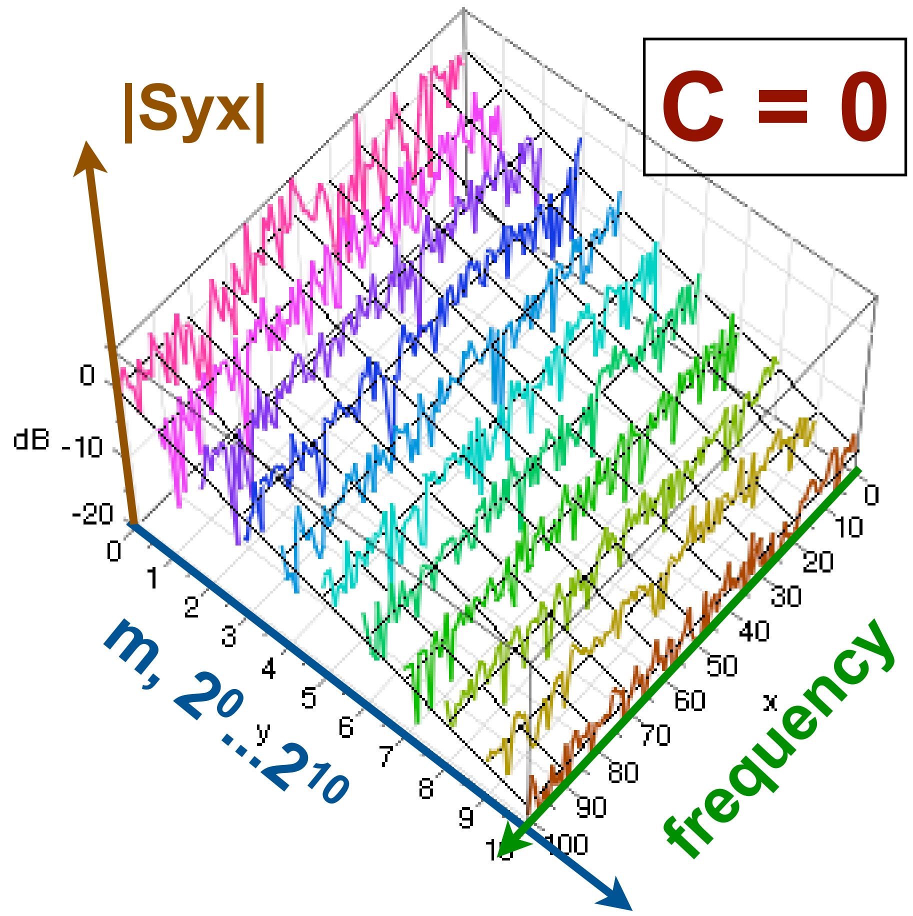
= √(0.182/m)

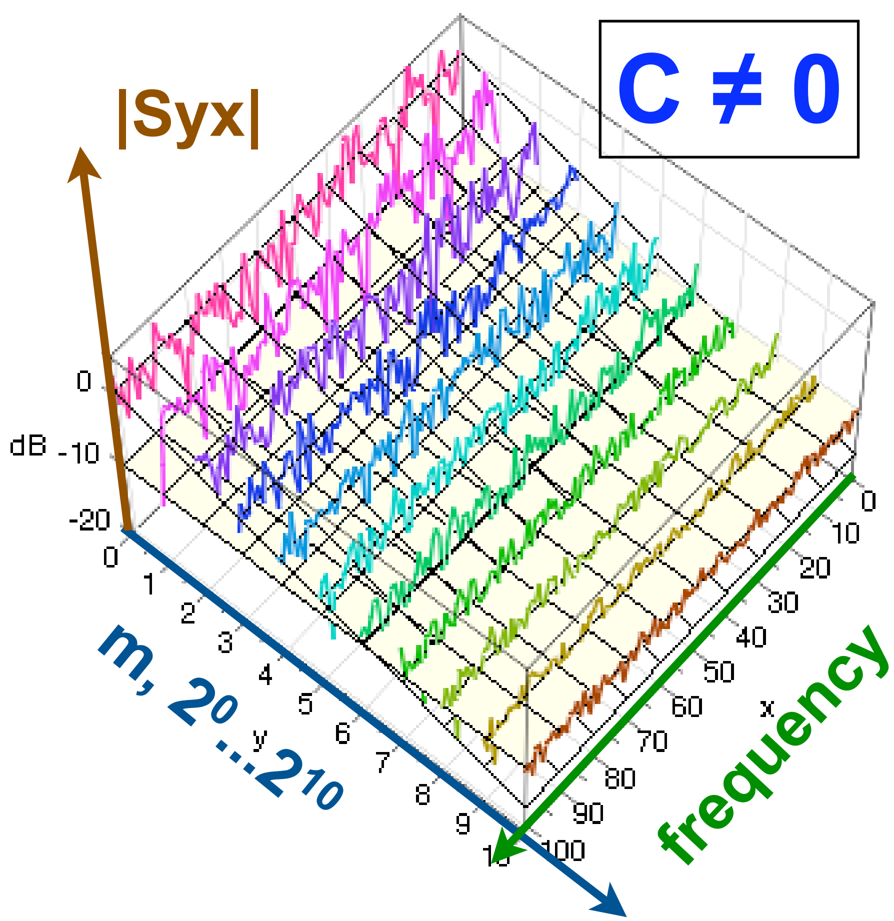
the dev / avg ratio is independent of m

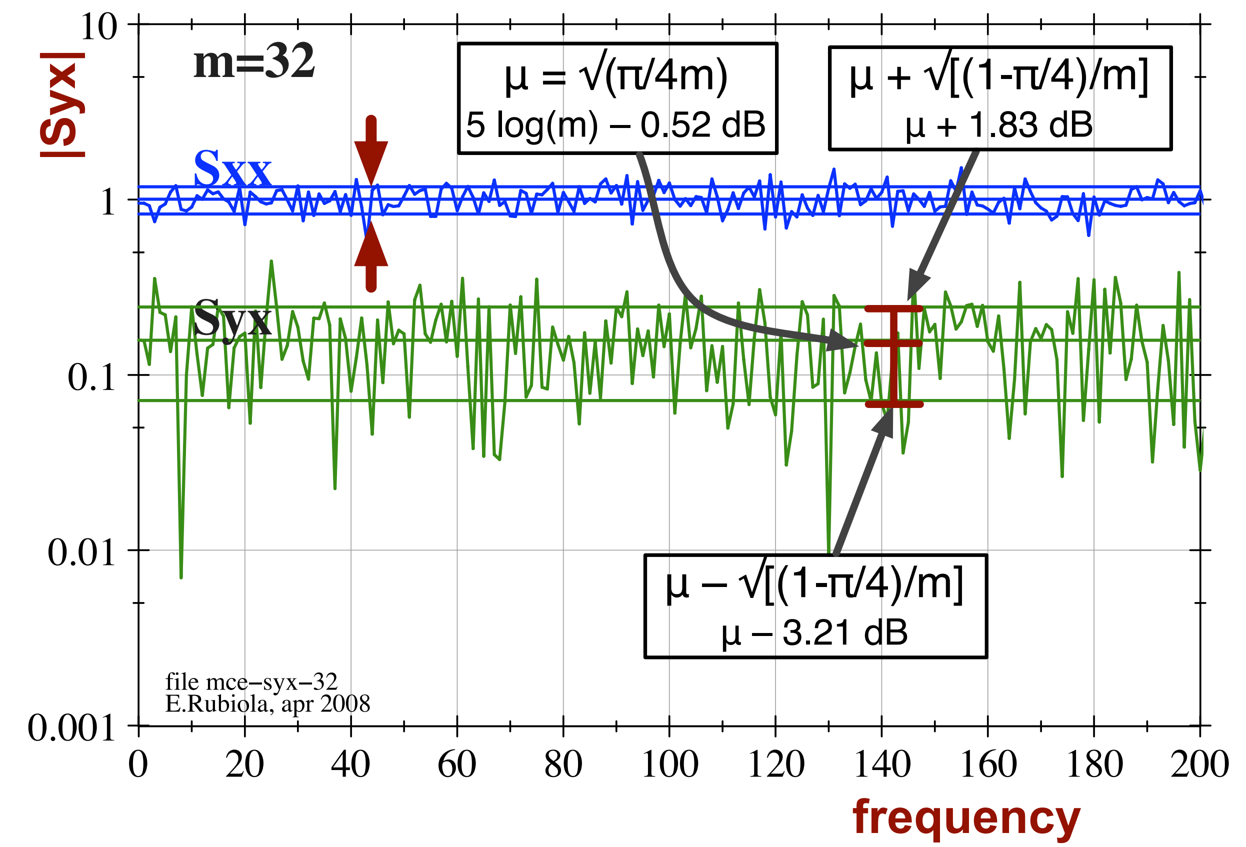
$$\frac{\sqrt{\mathbb{E}\{|S - \mathbb{E}\{S\}|^2\}}}{\mathbb{E}\{S\}} = \sqrt{\frac{\pi}{2} - 1}$$ = 0.756

The track thickness on the analyzer logarithmic scale is constant because the dev / avg ratio is independent of m

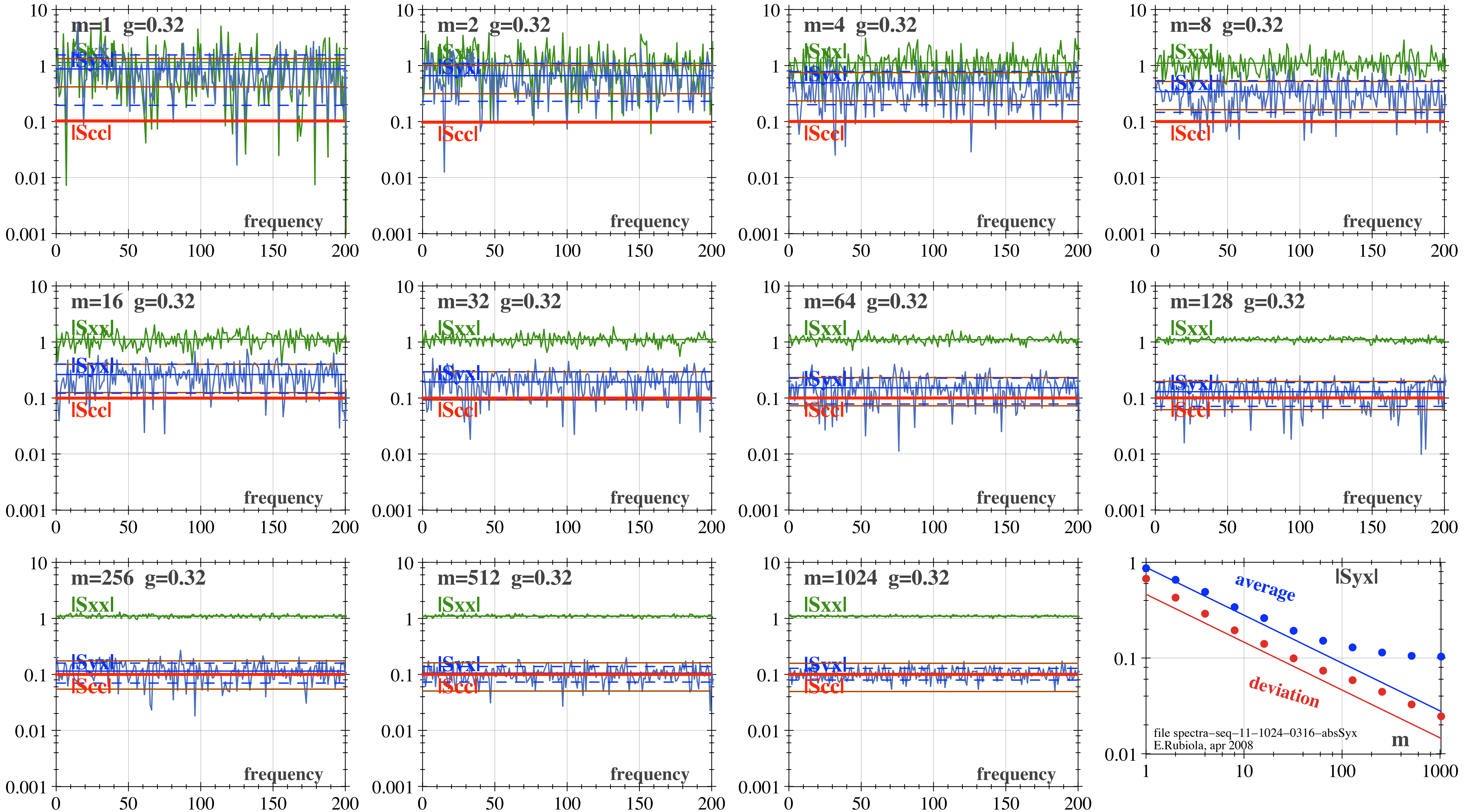
Example: Measurement of |Syx|





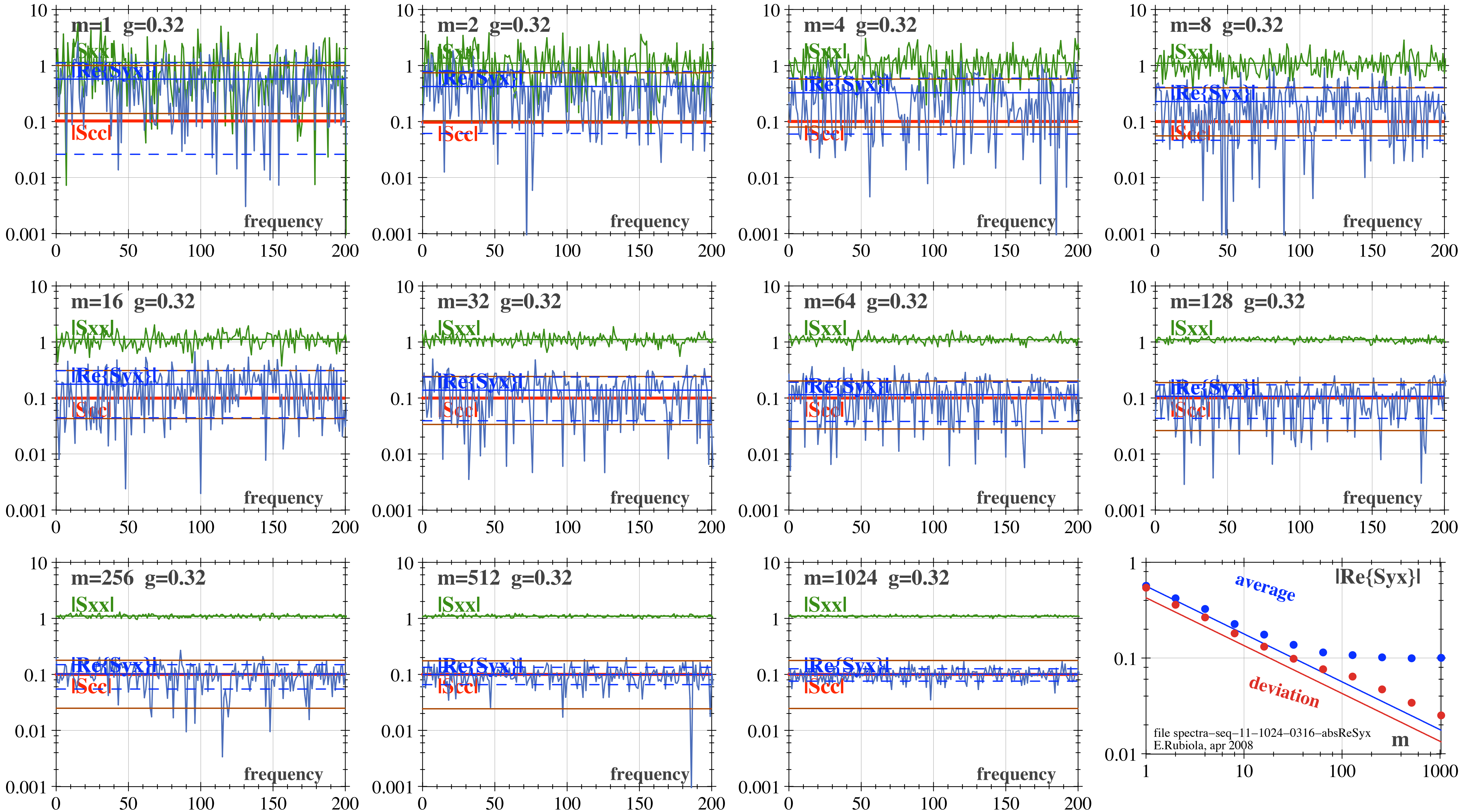


Measurement (C≠0), |Syx|



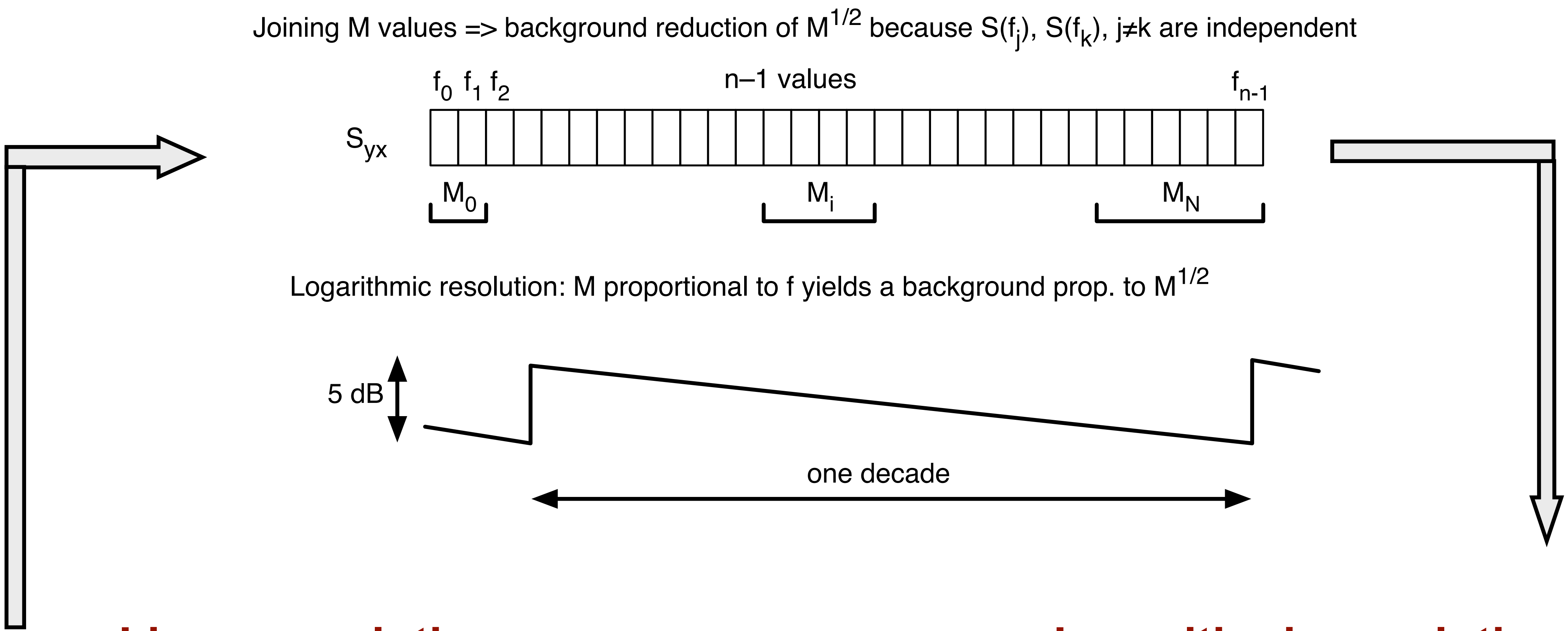
Running the measurement, m increases
S_{xx} shrinks => better confidence level
S_{yx} decreases => higher single-channel noise rejection

Measurement (C≠0), |Re{Syx}|



Running the measurement, m increases
S_{xx} shrinks => better confidence level
S_{yx} decreases => higher single-channel noise rejection

Linear vs. logarithmic resolution



Linear resolution

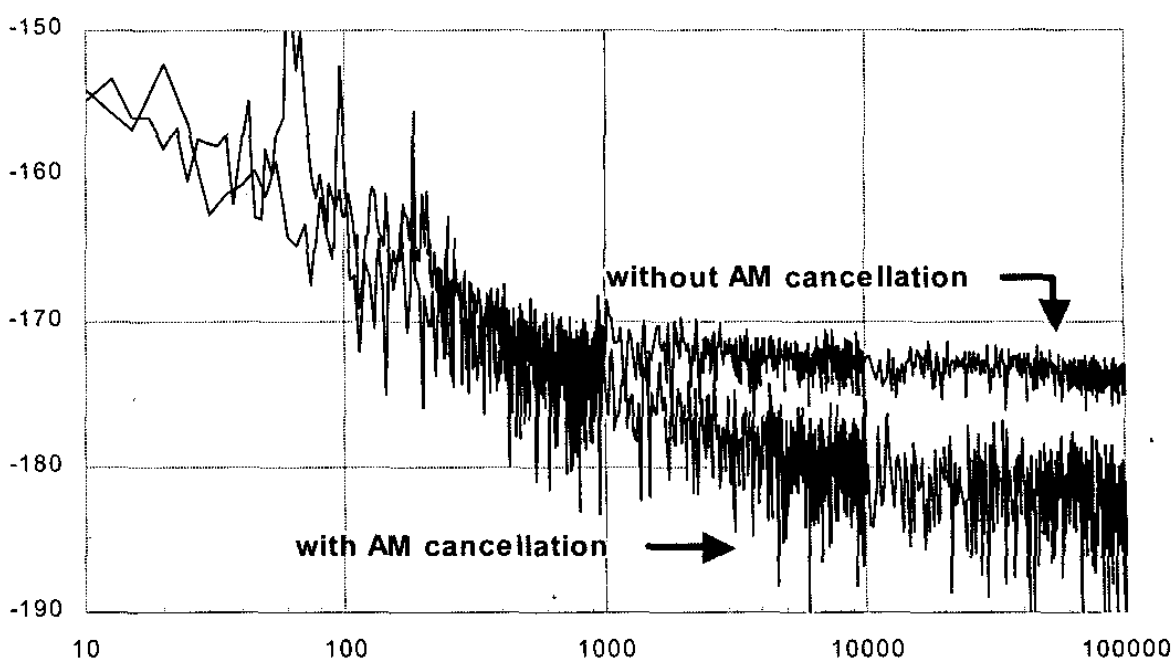


Fig.5, G. Cibiel, TUFFC 49(6) jun 2002

Logarithmic resolution

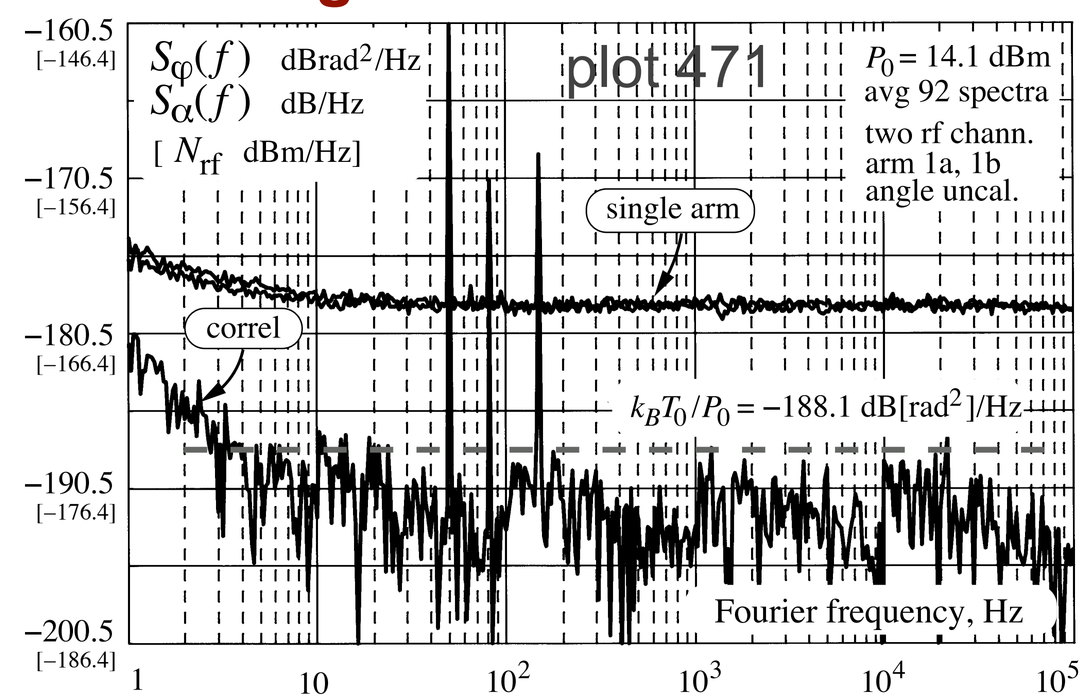


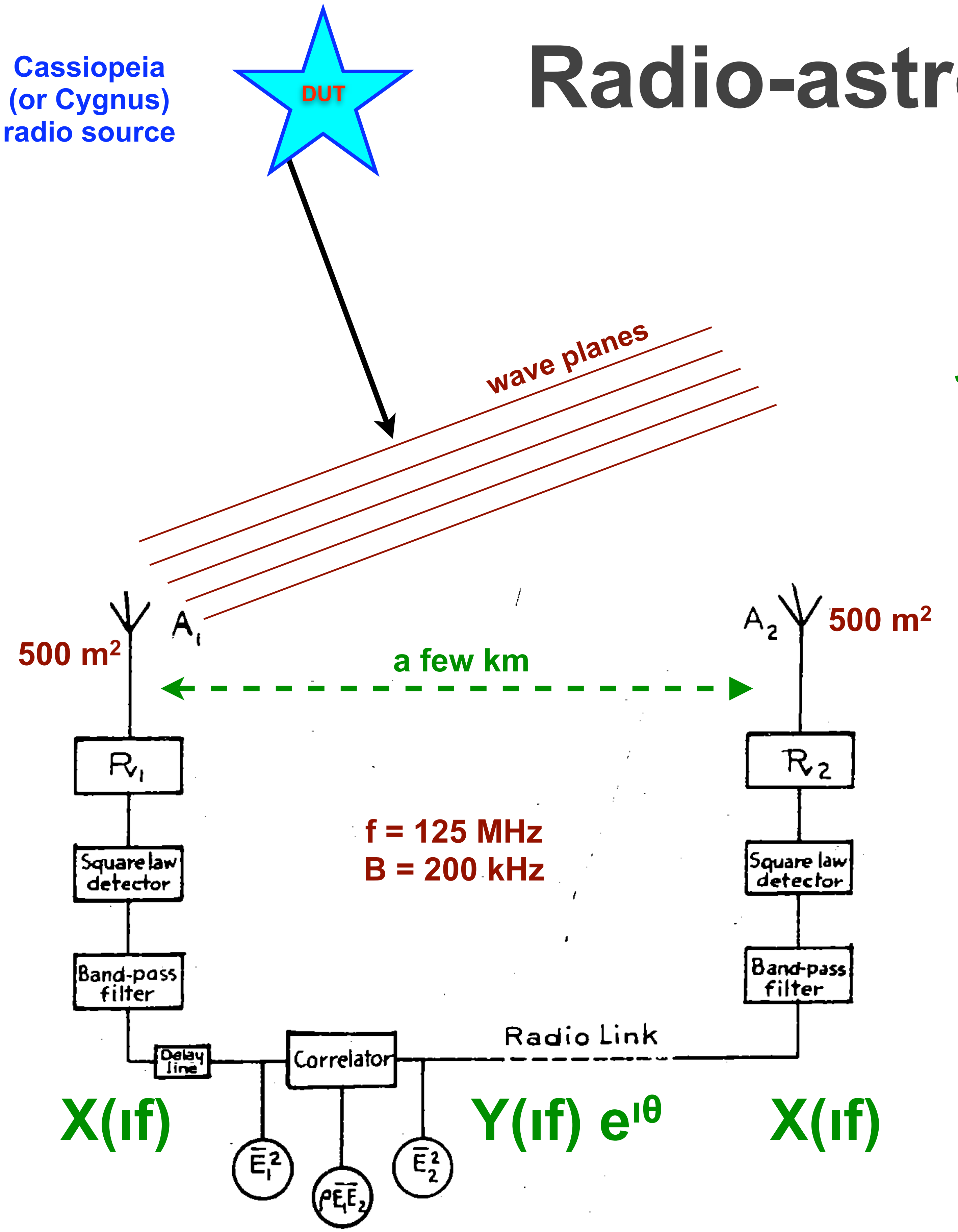
Fig.7, E. Rubiola, V. Giordano, RSI 73(6) jun 2002

Part 2 – Applications

Applications

- Radio-astronomy (Hanbury-Brown, 1952)
- Early implementations
- Radiometry (Allred, 1962)
- Noise calibration (Spietz, 2003)
- Frequency noise (Vessot 1964)
- Phase noise (Walls 1976)
- Phase noise (Lance, 1982)
- Phase noise (Rubiola 2000 & 2002))
- Effect of amplitude noise (Rubiola, 2007)
- Dual-mixer time-domain instrument (Allan 1975, Stein 1983)
- Amplitude noise & laser RIN (Rubiola 2006)
- Semiconductors (Sampietro RSI 1999)
- Electromigration in thin films (Stoll 1989)
- Fundamental definition of temperature
- Hanbury Brown - Twiss effect (Hanbury-Brown & Twiss 1956, Glattli 2004)

Radio-astronomy



Measurement of the apparent angular size of stellar radio sources

Jodrell Bank, Manchester

- The radio link breaks the hypothesis of symmetry of the two channels, introducing a phase θ
- The cross spectrum is complex
- The the antenna directivity results from the phase relationships
- The phase of the cross spectrum indicates the direction of the radio source

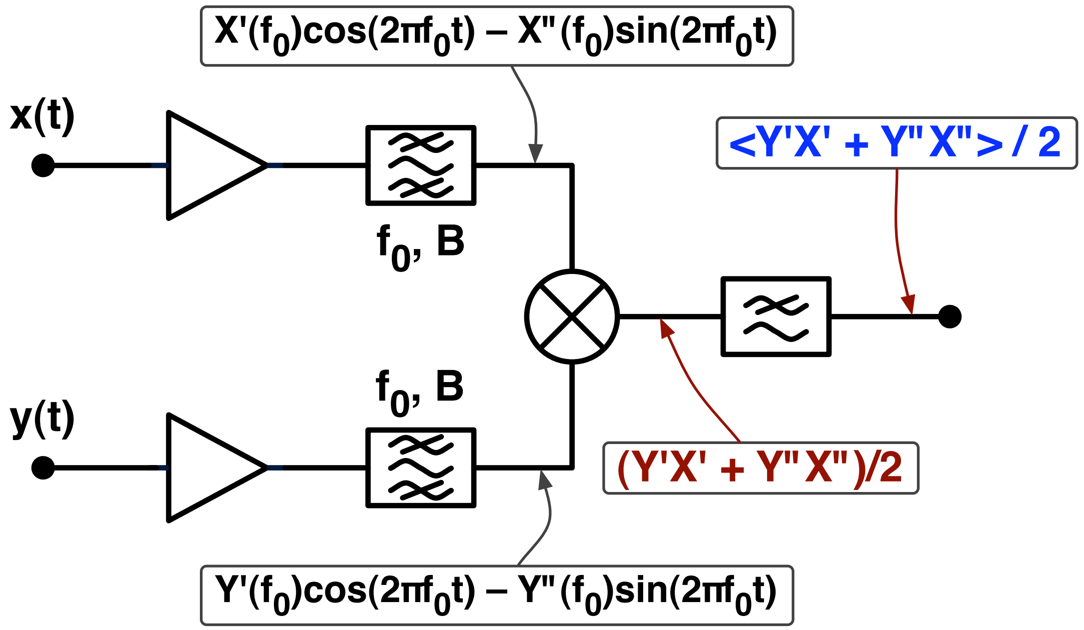
R. Hanbury Brown & al., Nature 170(4338) p.1061-1063, 20 Dec 1952

R. Hanbury Brown, R. Q. Twiss, Phyl. Mag. ser.7 no.366 p.663-682

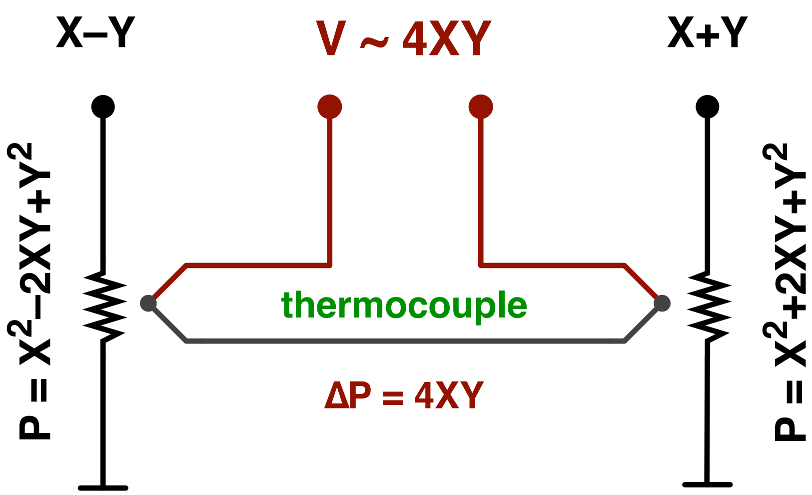
Early implementations

1940-1950 technology

Analog correlator



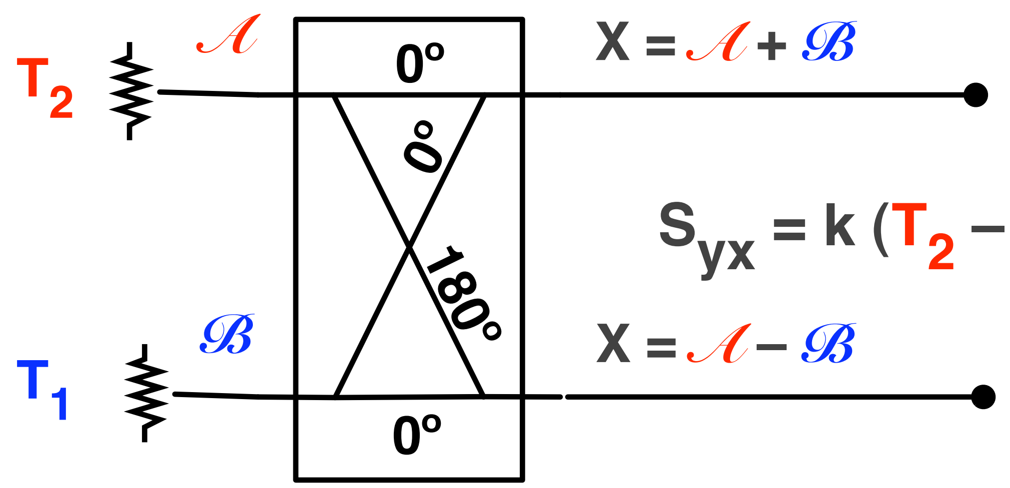
Analog multiplier



Spectral analysis at the single frequency f_0, in the bandwidth B
Need a filter pair for each Fourier frequency

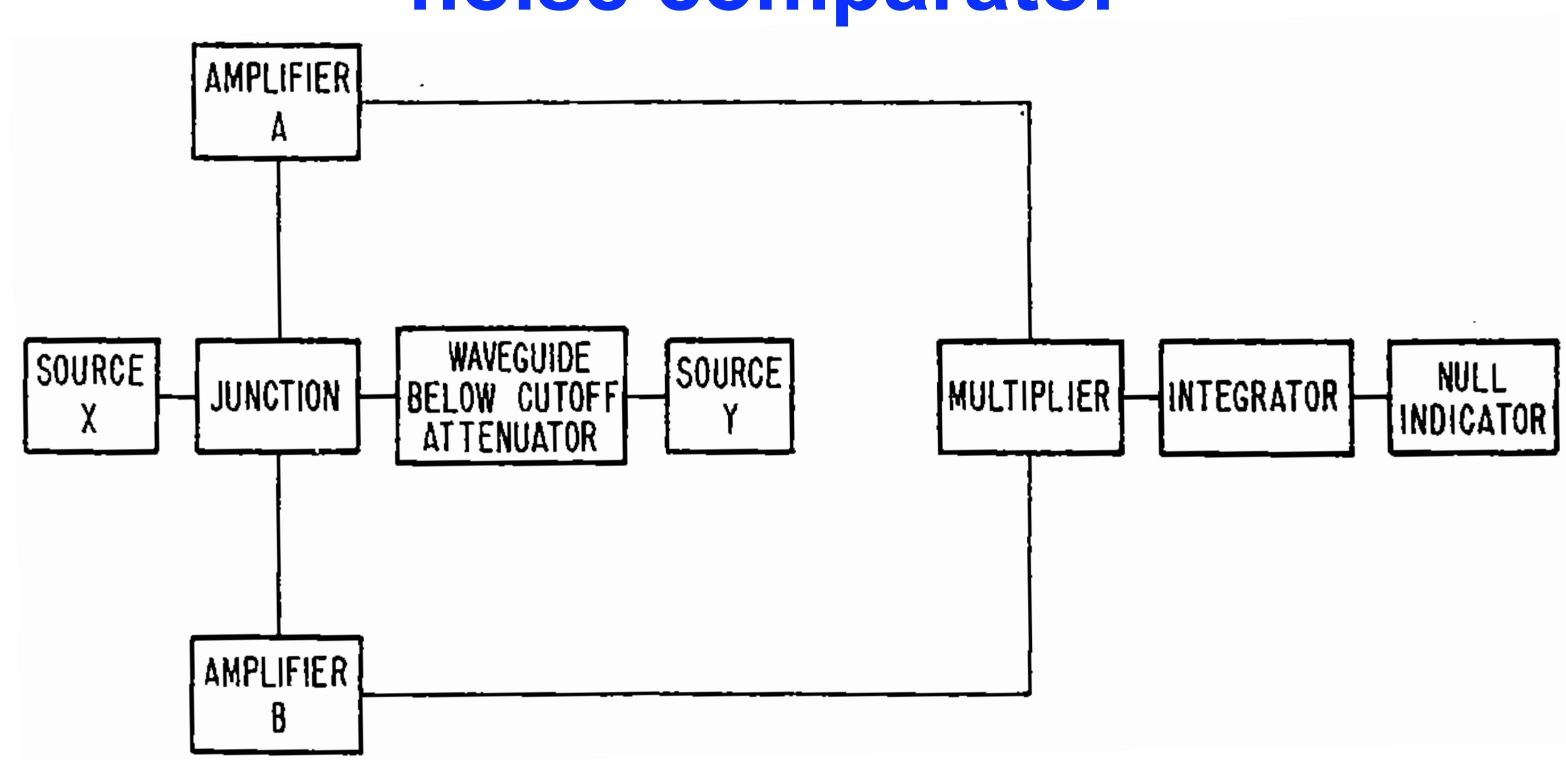
Radiometry

correlation and anti-correlation



$$S_{yx} = k\,(T_2 - T_1)\,/\,2$$

noise comparator



C. M. Allred, A precision noise spectral density comparator, J. Res. NBS 66C no.4 p.323-330, Oct-Dec 1962

Noise calibration

thermal noise $S = kT$

shot noise $S = 2qI_{avg}R$

high accuracy of I_{avg} with a dc instrument

Compare shot and thermal noise with a noise bridge

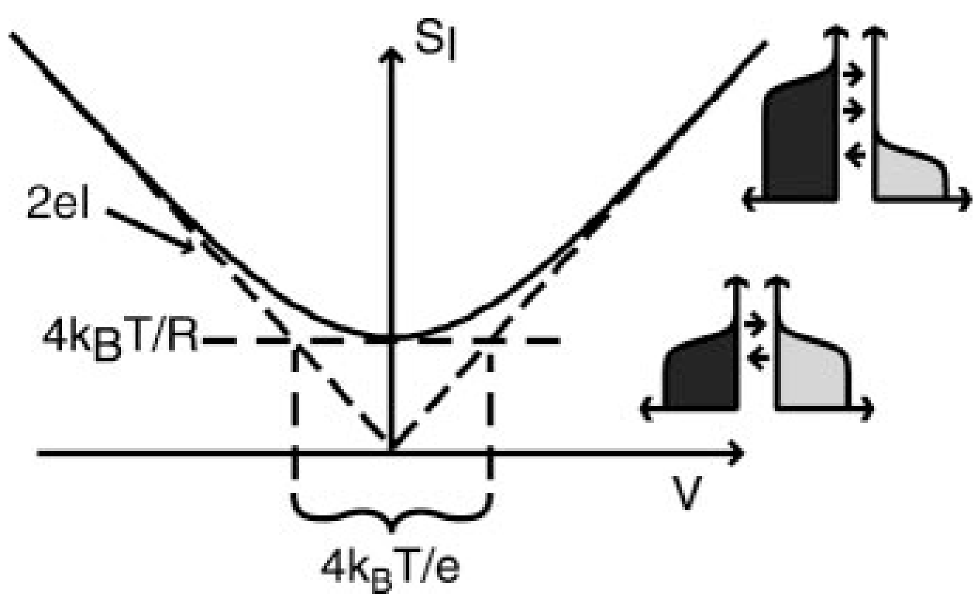


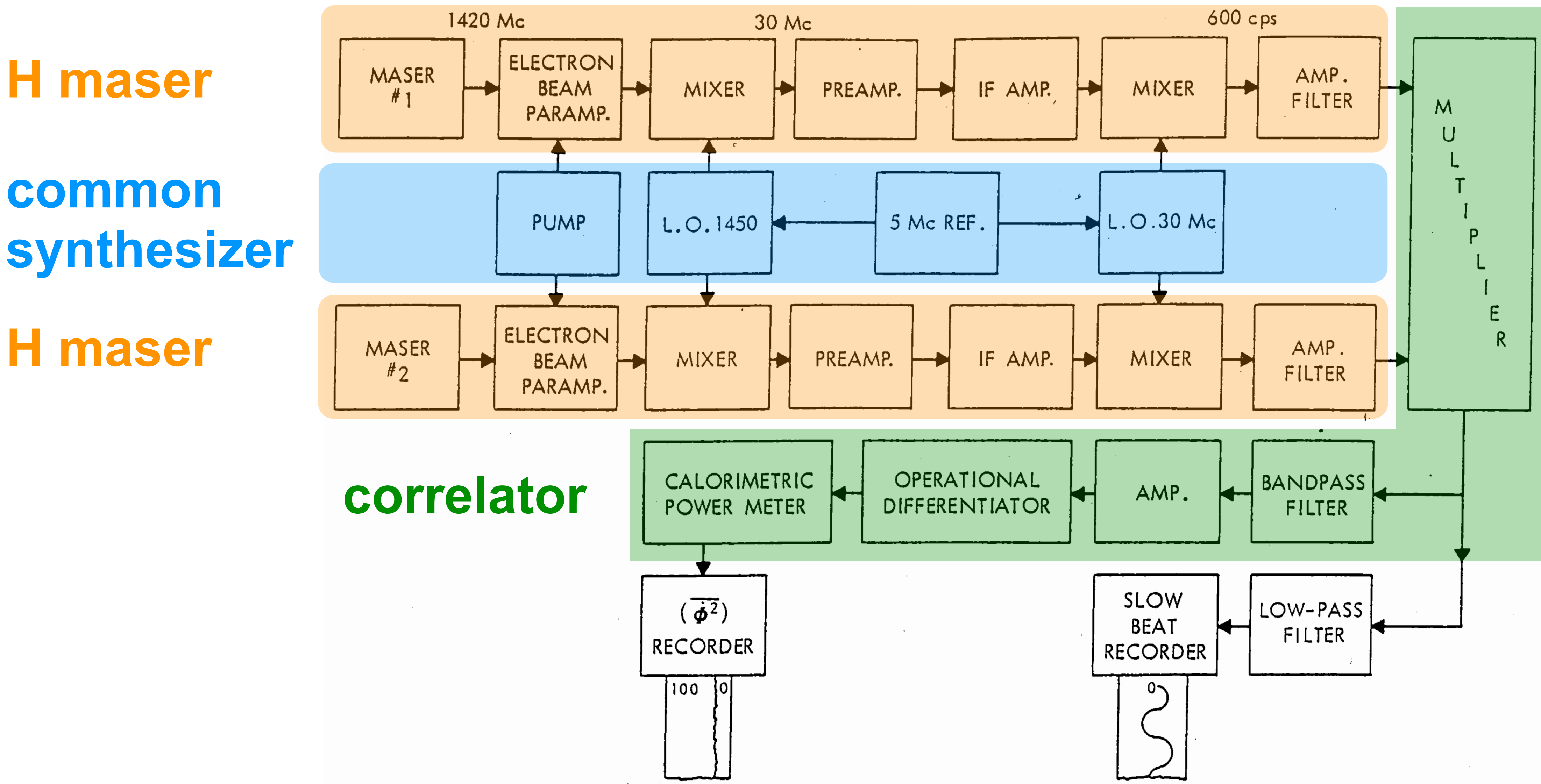
Fig. 1. Theoretical plot of current spectral density of a tunnel junction (Eq. 3) as a function of dc bias voltage. The diagonal dashed lines indicate the shot noise limit, and the horizontal dashed line indicates the Johnson noise limit. The voltage span of the intersection of these limits is $4k_BT/e$ and is indicated by vertical dashed lines. The bottom inset depicts the occupancies of the states in the electrodes in the equilibrium case, and the top inset depicts the out-of-equilibrium case where $eV \gg k_BT$.

This idea could turn into a re-definition of the temperature

In a tunnel junction, theory predicts the amount of shot and thermal noise

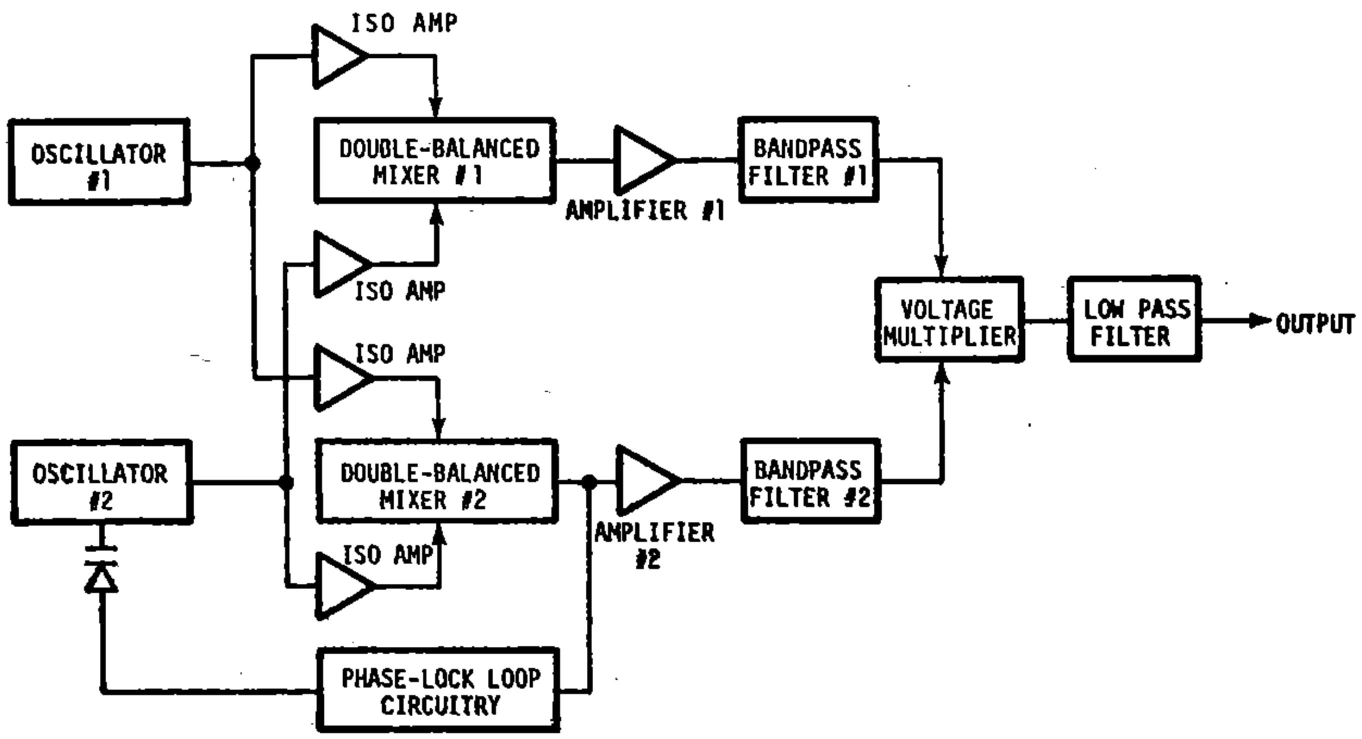
L. Spietz & al., Primary electronic thermometry using the shot noise of a tunnel junction, Science 300(20) p. 1929-1932, jun 2003

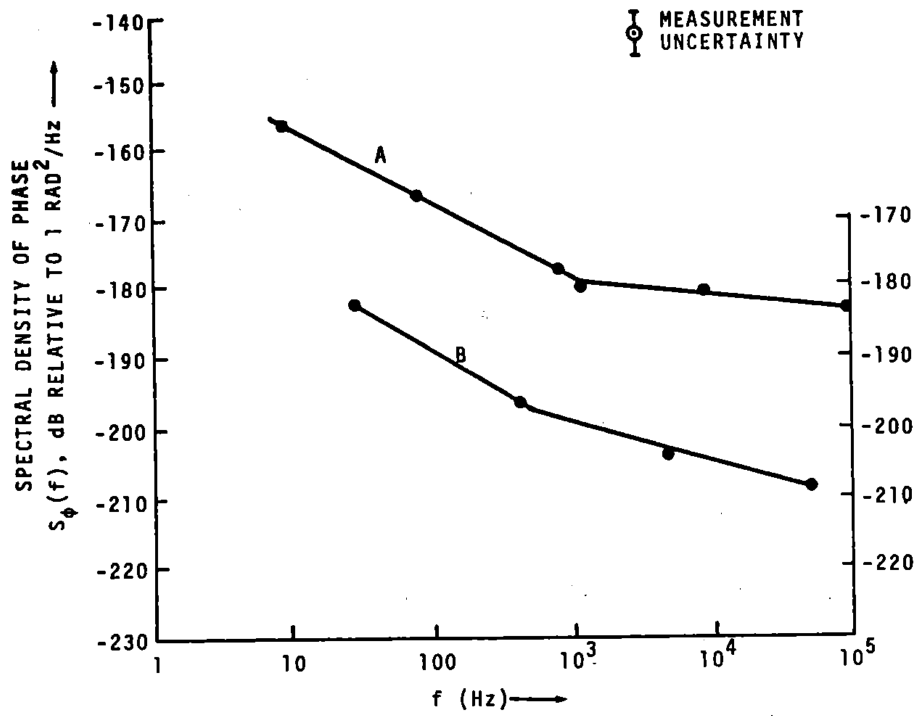
Measurement of H-maser frequency noise



R. F. C. Vessot, Proc. Nasa Symp. on Short Term Frequency Stability p.111-118, Greenbelt, MD, 23-24 Nov 1964

Phase noise measurement

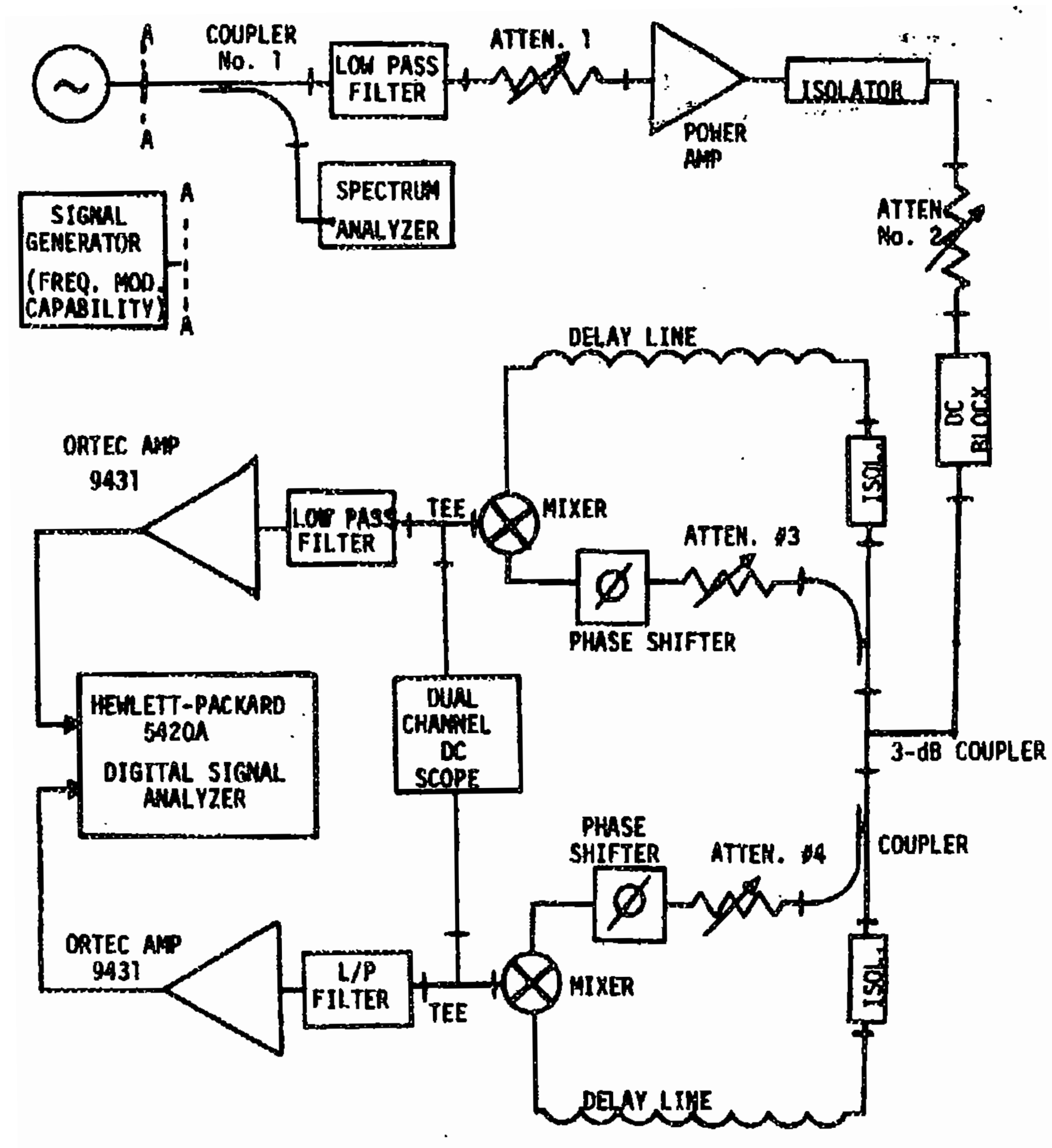




(relatively) large correlation bandwidth provides low noise floor in a reasonable time

F.L. Walls & al, Proc. 30th FCS pp.269-274, 1976
popular after W. Walls, Proc. 46th FCS pp.257-261, 1992

Oscillator phase noise measurement



Original idea:
D. Halford's NBS notebook
F10 p.19-38, apr 1975

First published: A. L. Lance & al, CPEM Digest, 1978

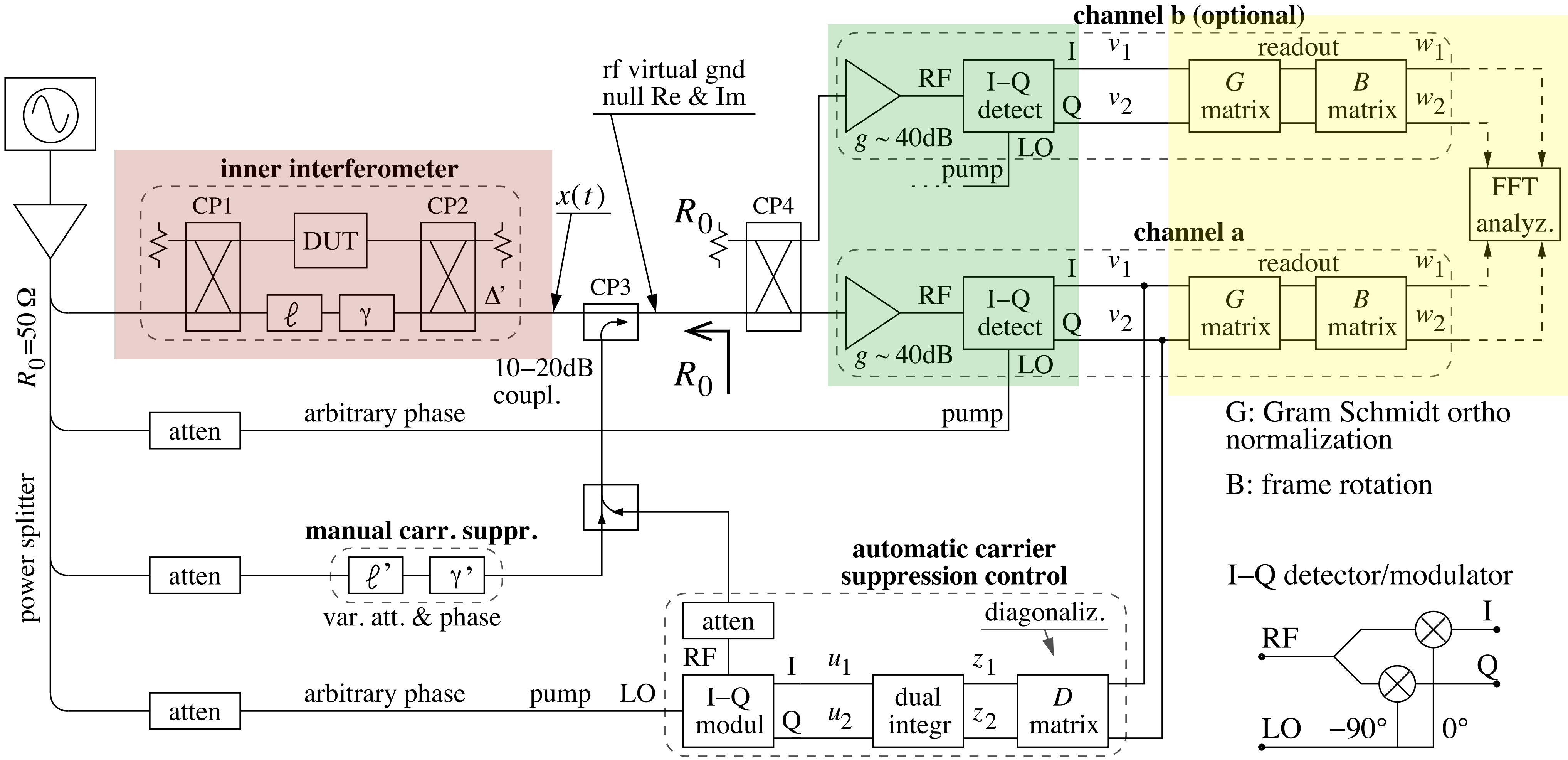
The delay line converts the frequency noise into phase noise

The high loss of the coaxial cable limits the maximum delay

Updated version:
The optical fiber provides long delay with low attenuation (0.2 dB/km or 0.04 dB/μs)

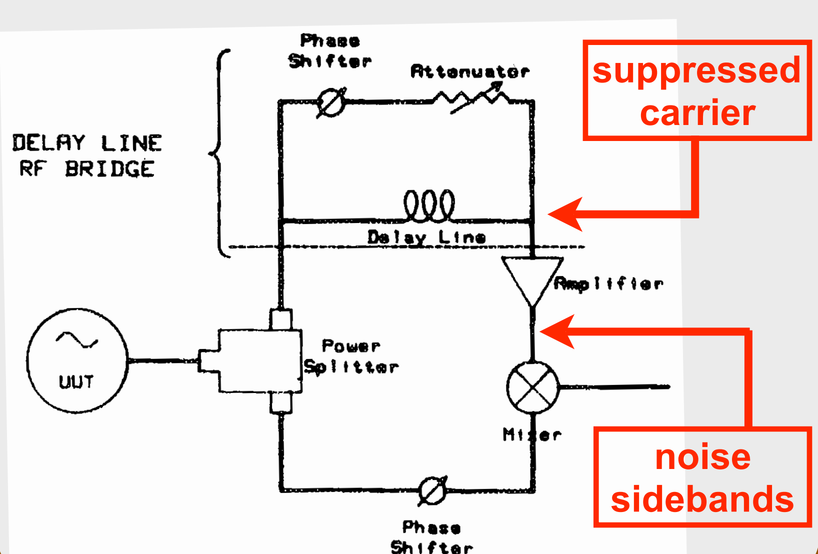
A.L. Lance, W.D. Seal, F. Labaar ISA Transact.21 (4) p.37-84, Apr 1982

Phase noise measurement

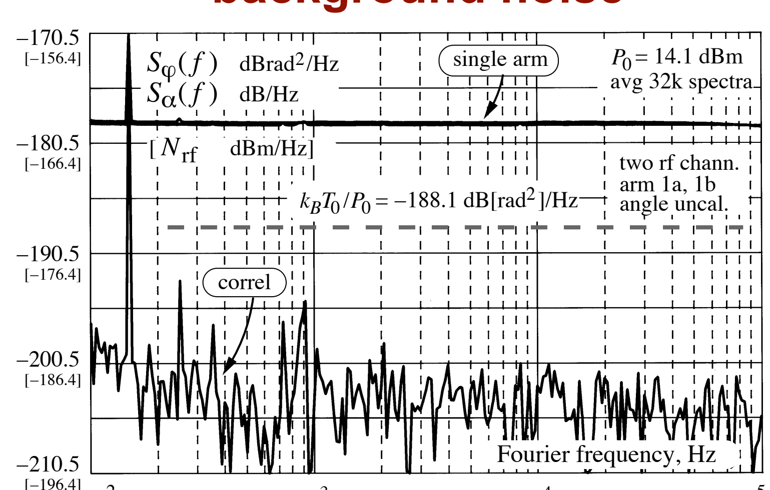


A.L. Lance & al., ISA Transact. 2(4) p.37-84 apr 1982

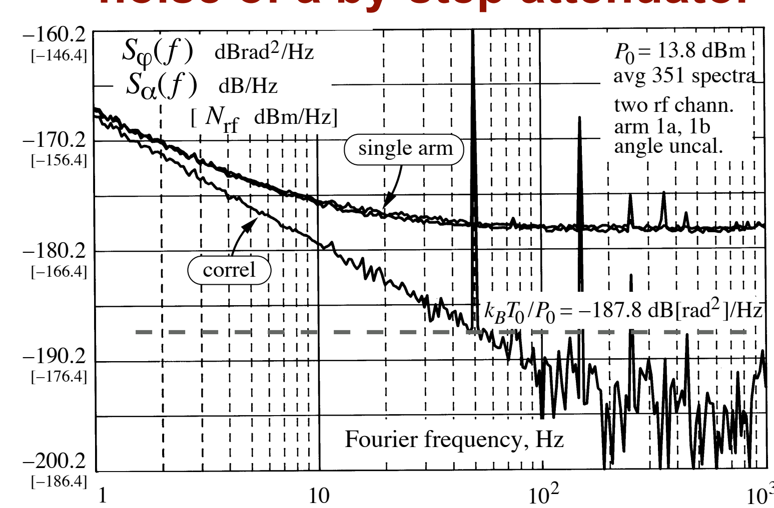
F. Labaar, Microwaves 21(3) p.65-69, mar 1982



background noise



noise of a by-step attenuator

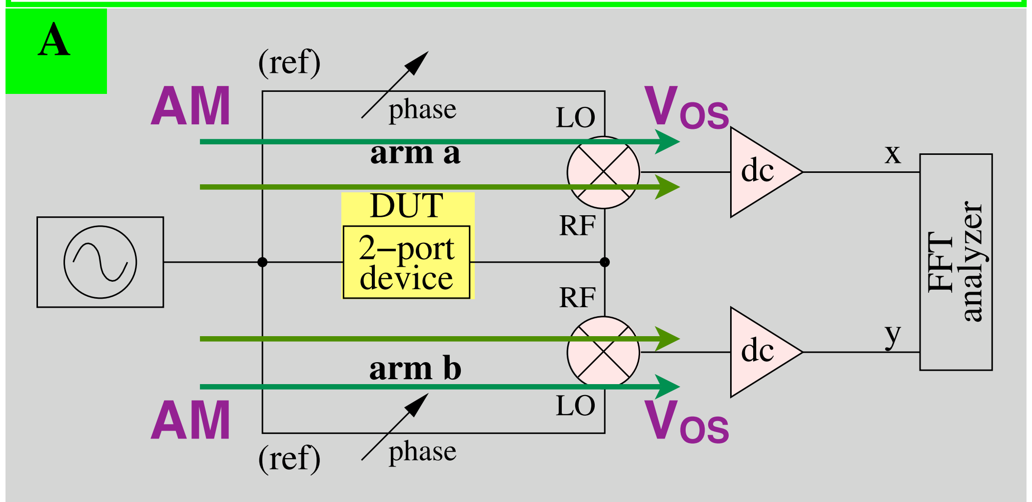


E. Rubiola, V. Giordano, Rev. Sci. Instrum. 71(8) p.3085-3091, aug 2000

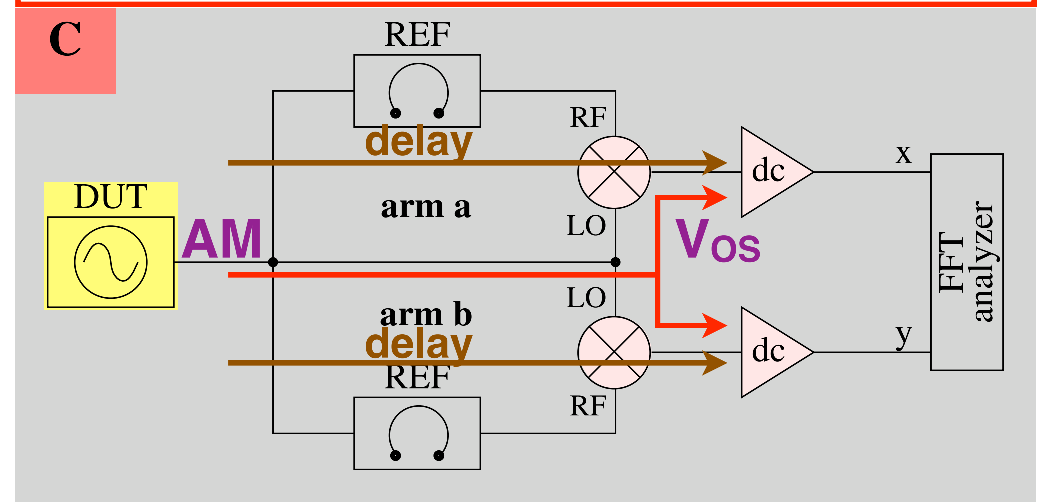
E. Rubiola, V. Giordano, Rev. Sci. Instrum. 73(6) pp.2445-2457, jun 2002

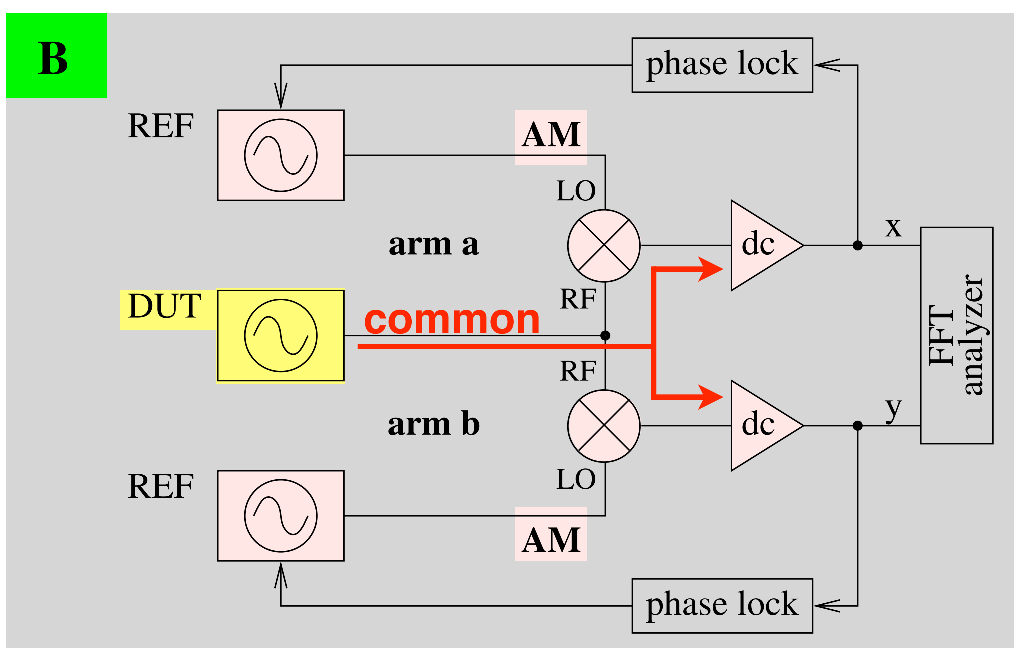
Effect of amplitude noise

Should set both channels at the sweet point, if exists

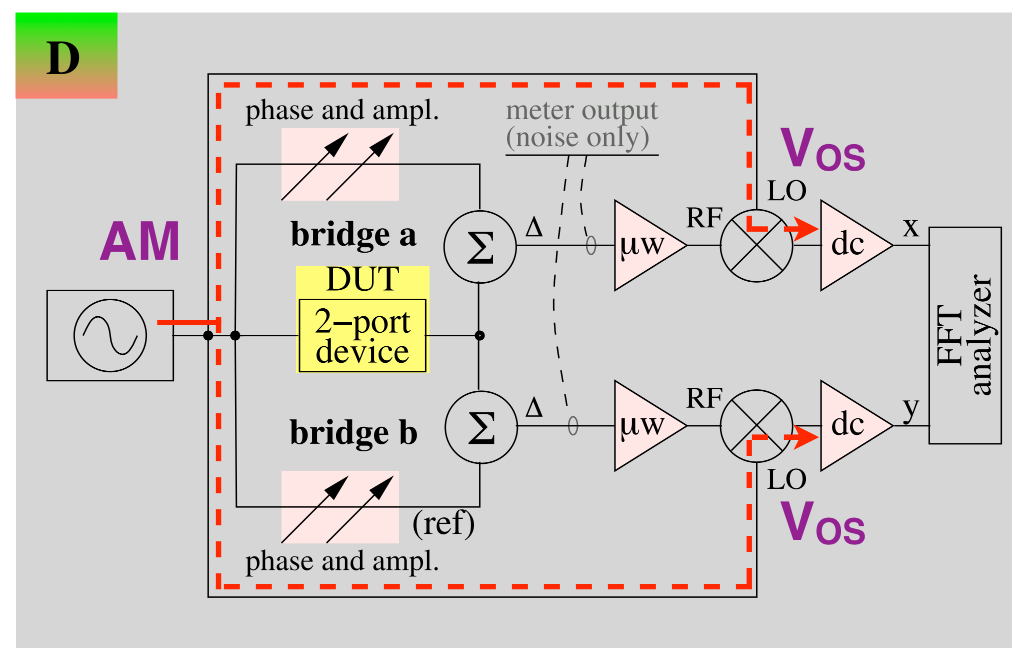


The delay de-correlates the two inputs, so there is no sweet point





Should set both channels at the sweet point of the RF input, if exists, by offsetting the PLL or by biasing the IF

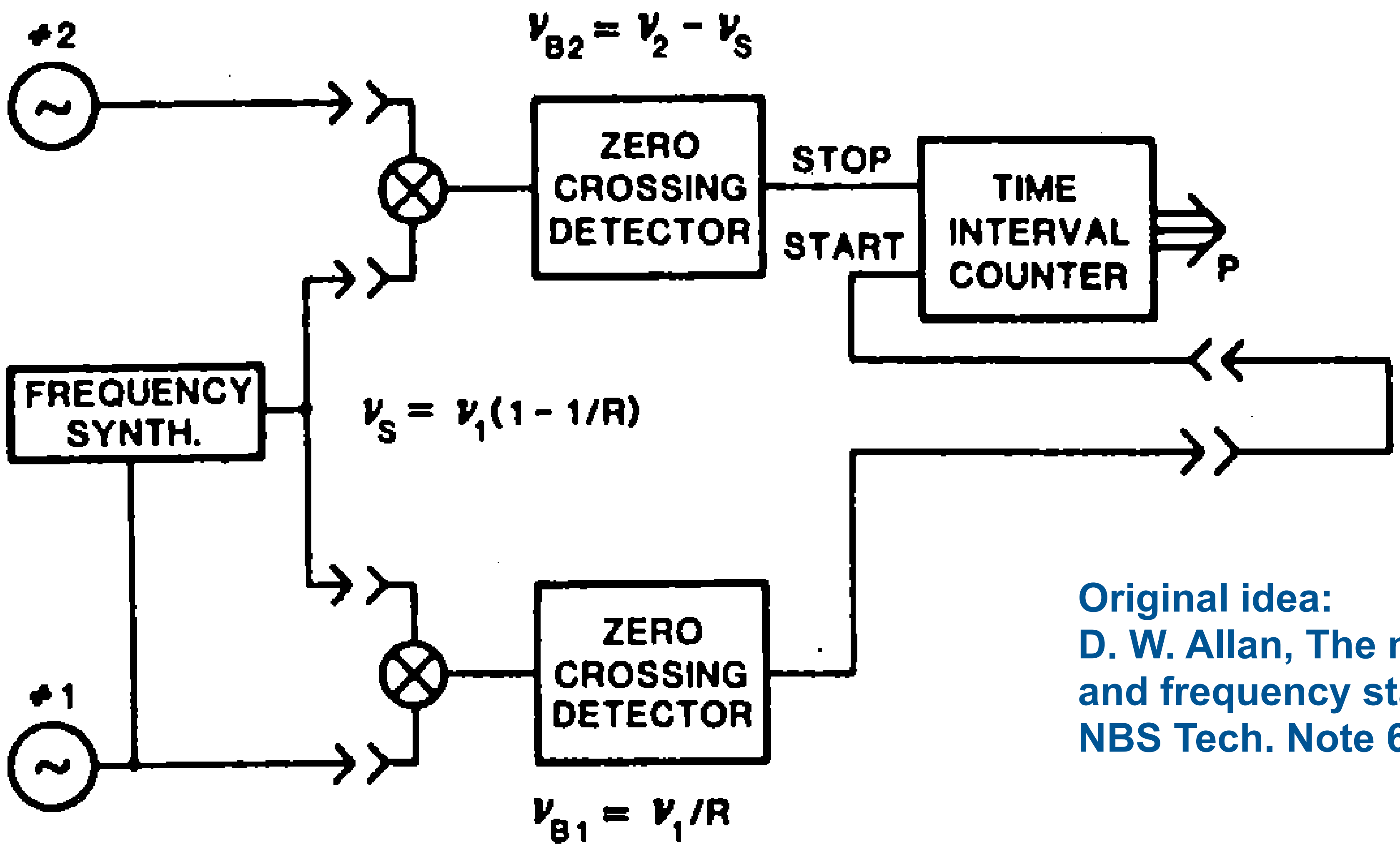


The effect of the AM noise is strongly reduced by the RF amplification

pink: noise rejected by correlation and averaging

E. Rubiola, R. Boudot, IEEE Transact. UFFC 54(5) pp.926-932, may 2007

Dual-mixer time-domain instrument



Original idea:
D. W. Allan, The measurement of frequency and frequency stability of precision oscillators, NBS Tech. Note 669, 1975

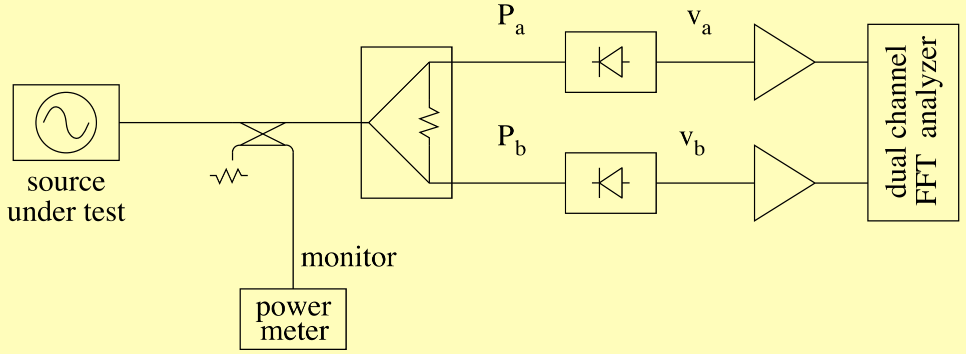
The average process rejects the mixer noise

This scheme is equivalent to the correlation method

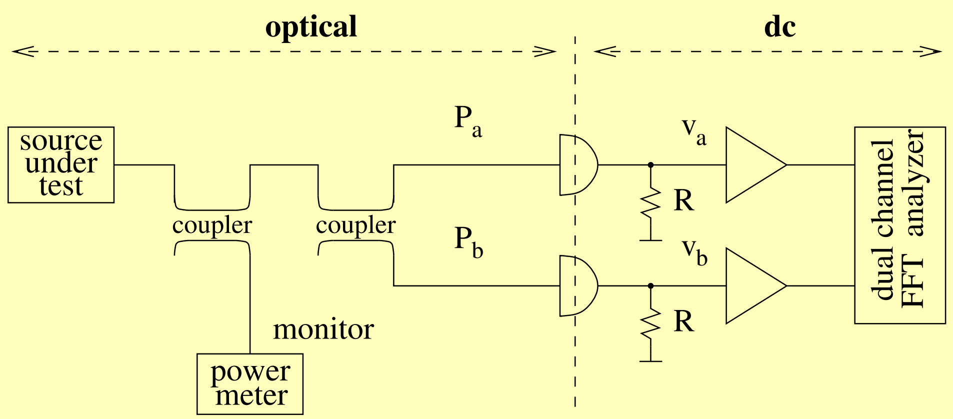
S. Stein & al., IEEE Transact. IM 32(1) p.227-230, mar 1983

Amplitude noise & laser RIN

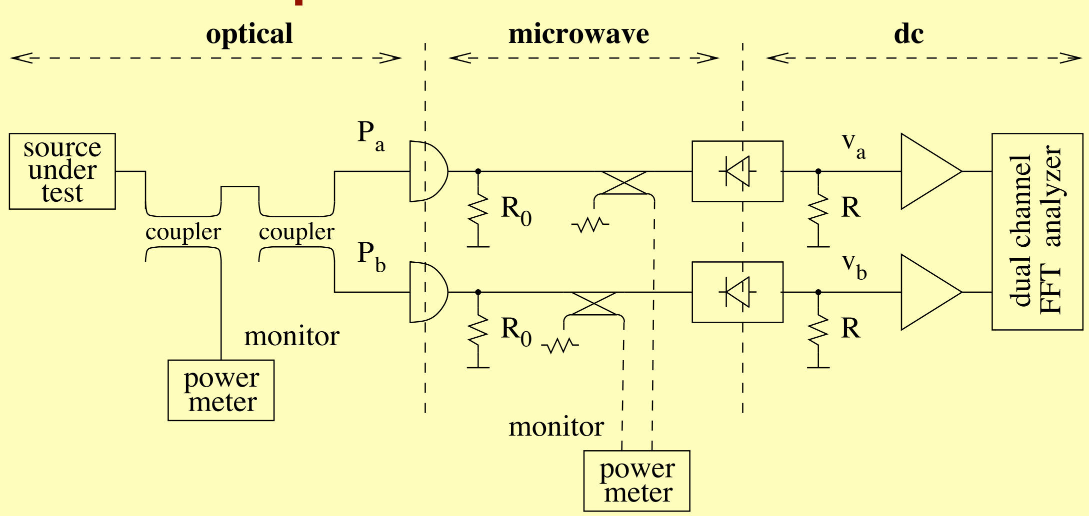
AM noise of RF/microwave sources



Laser RIN

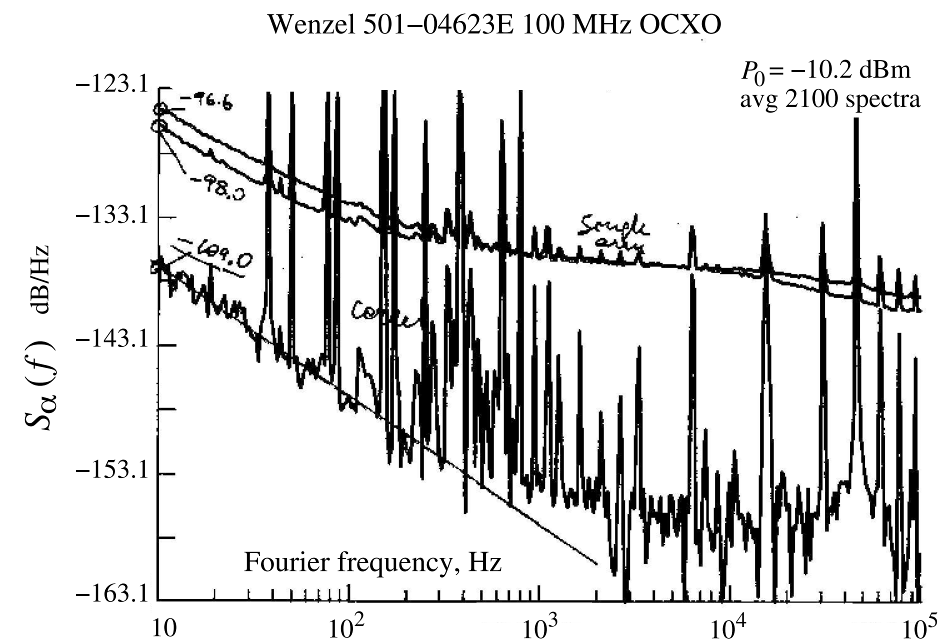


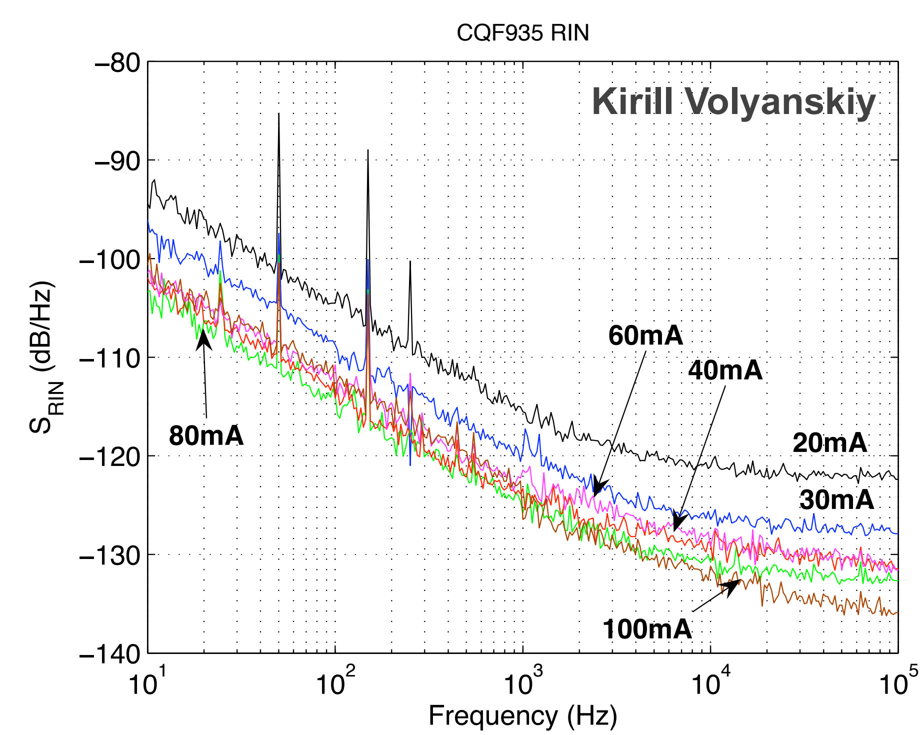
AM noise of photonic RF/microwave sources



E. Rubiola, the measurement of AM noise, dec 1995
arXiv:physics/0512082v1 [physics.ins-det]

- **In PM noise measurements, one can validate the instrument by feeding the same signal into the phase detector**
- **In AM noise this is *not possible* without a lower-noise reference**
- **Provided the crosstalk was measured otherwise, correlation enables to validate the instrument**





Measurement of noise in semiconductors

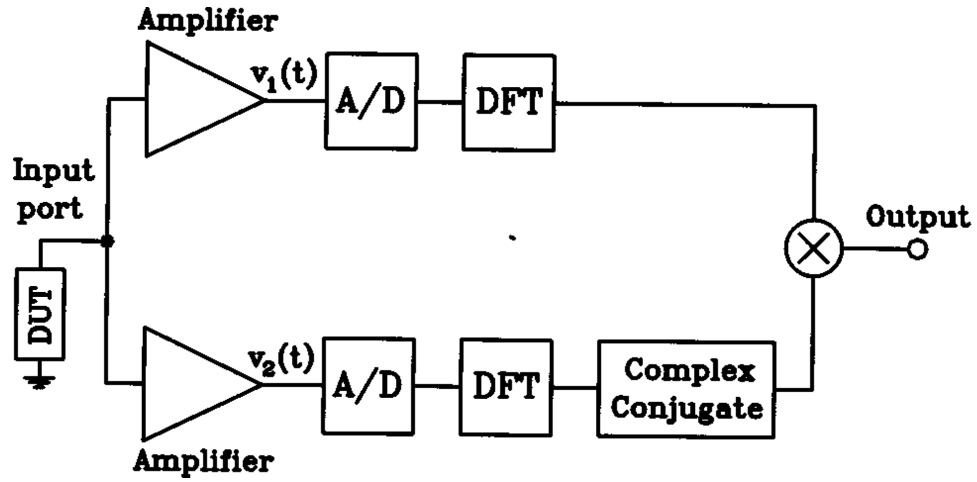


FIG. 2. Schematics of the building blocks of our correlation spectrum analyzer performing the suppression of the uncorrelated input noises by a digital processing of sampled data.

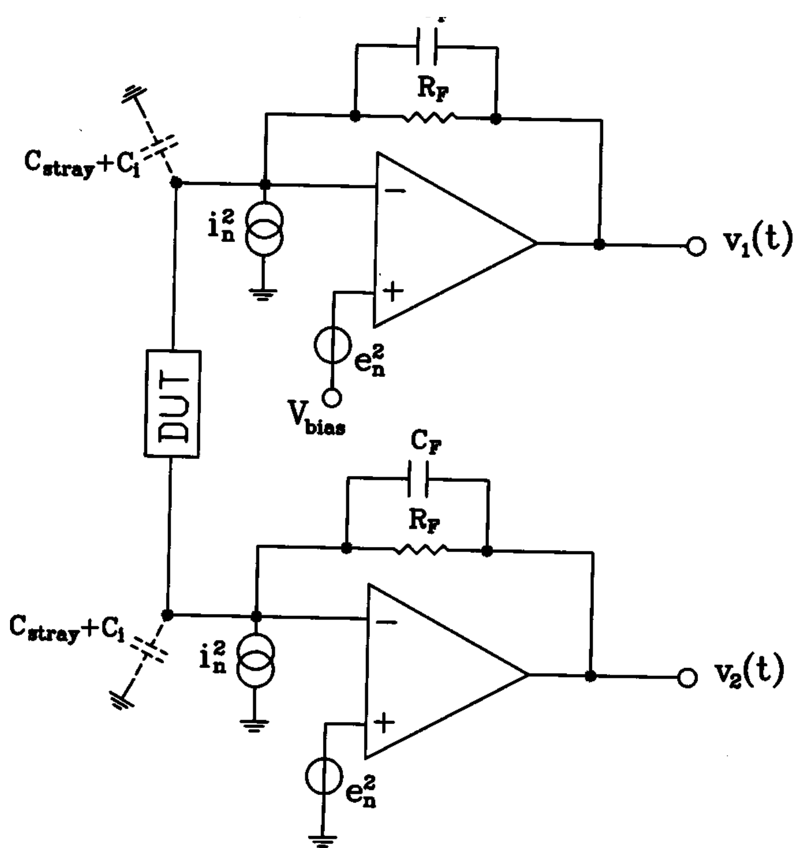


FIG. 3. Schematics of the active test fixture for current noise measurements.

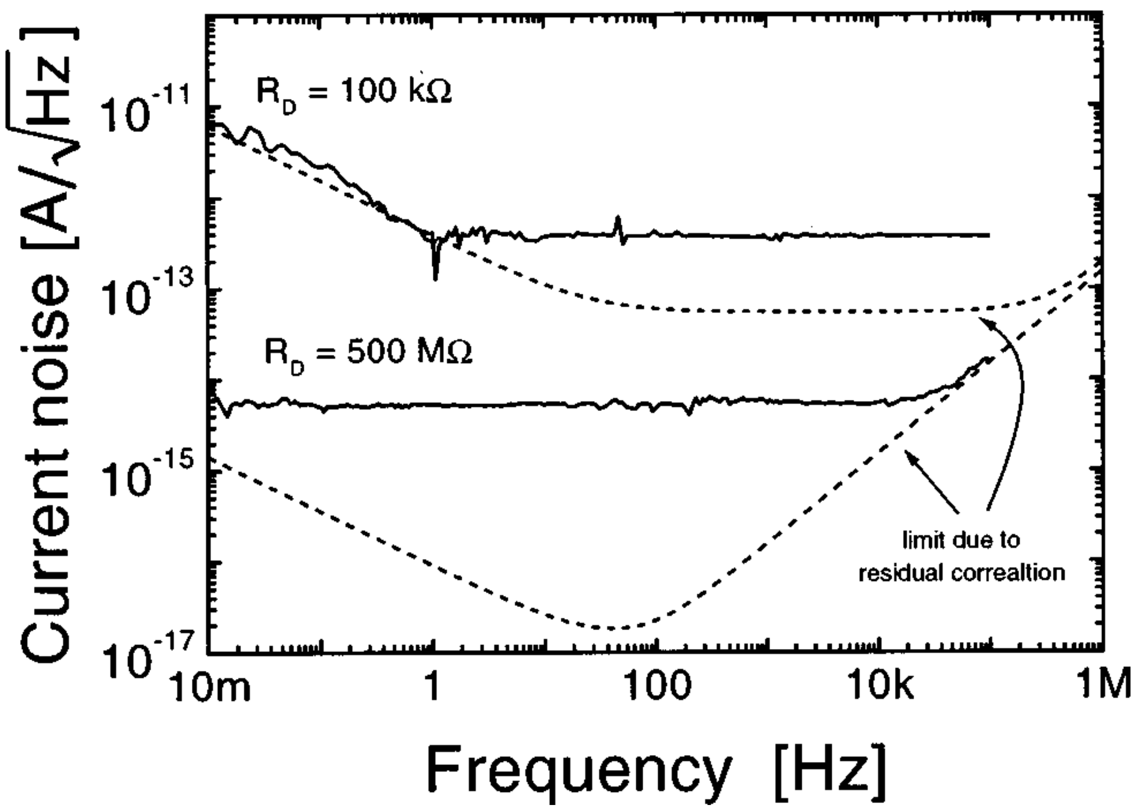
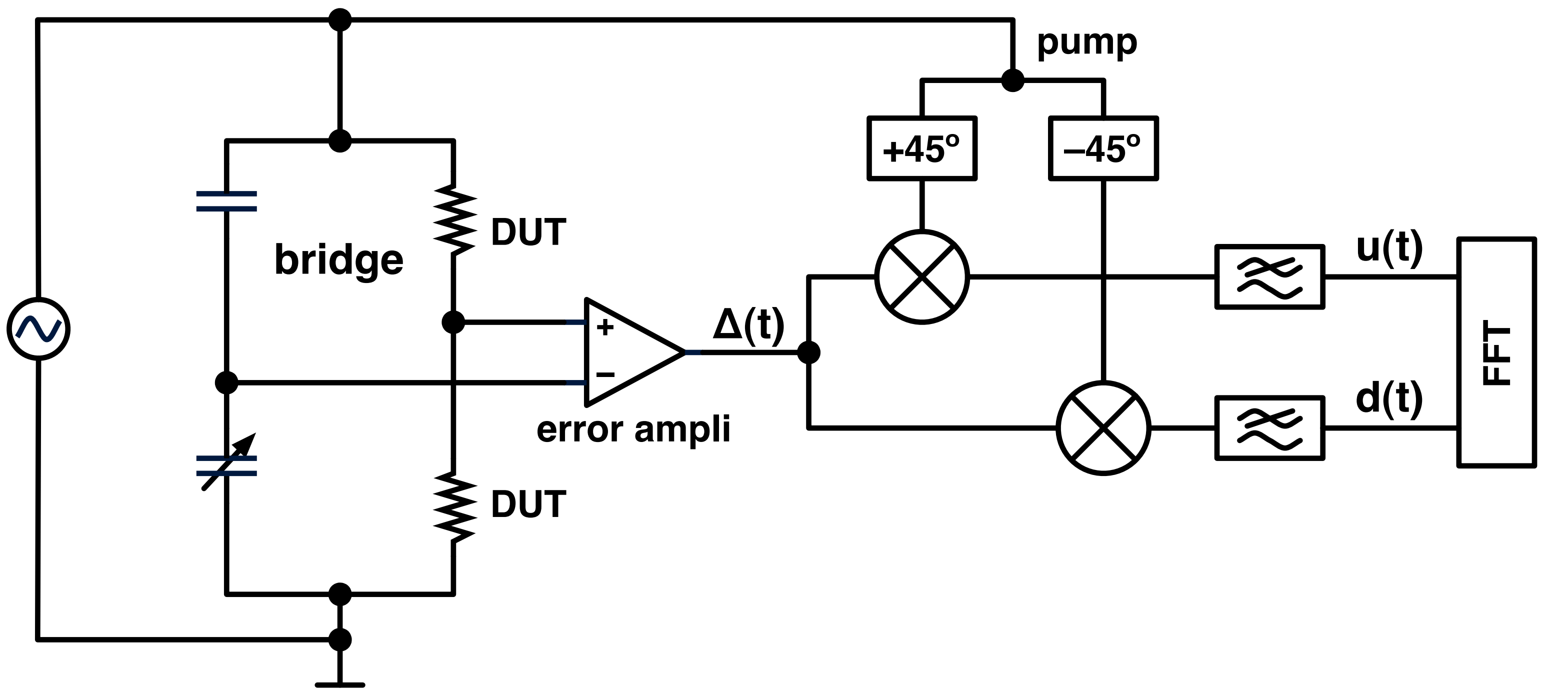


FIG. 9. Experimental frequency spectrum of the current noise from DUT resistances of 100 kΩ and 500 MΩ (continuous line) compared with the limits (dashed line) given by the instrument and set by residual correlated noise components.

M. Sampietro & al, Rev. Sci. Instrum 70(5) p.2520-2525, may 1999

Electromigration in thin films



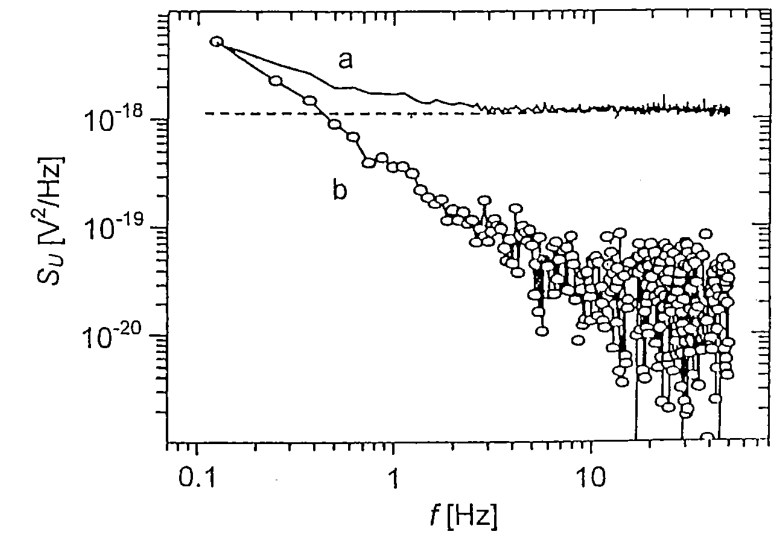
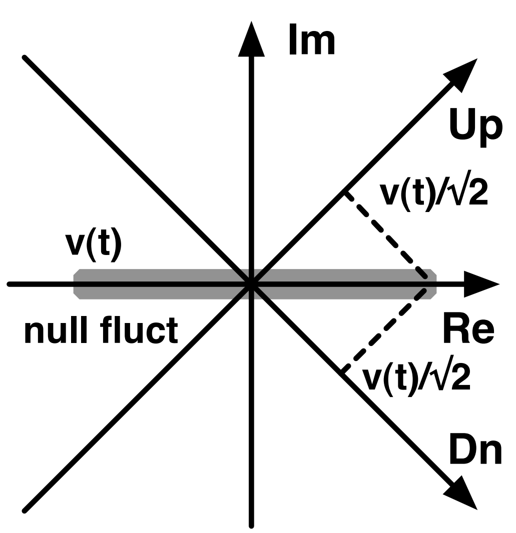


Fig. 1 1/f noise of an $AlSi_{0.01}Cu_{0.002}$ thin film measured at room temperature (a) without and (b) with the phase-sensitive ac correlation technique. The Johnson noise level is indicated by the dashed line.

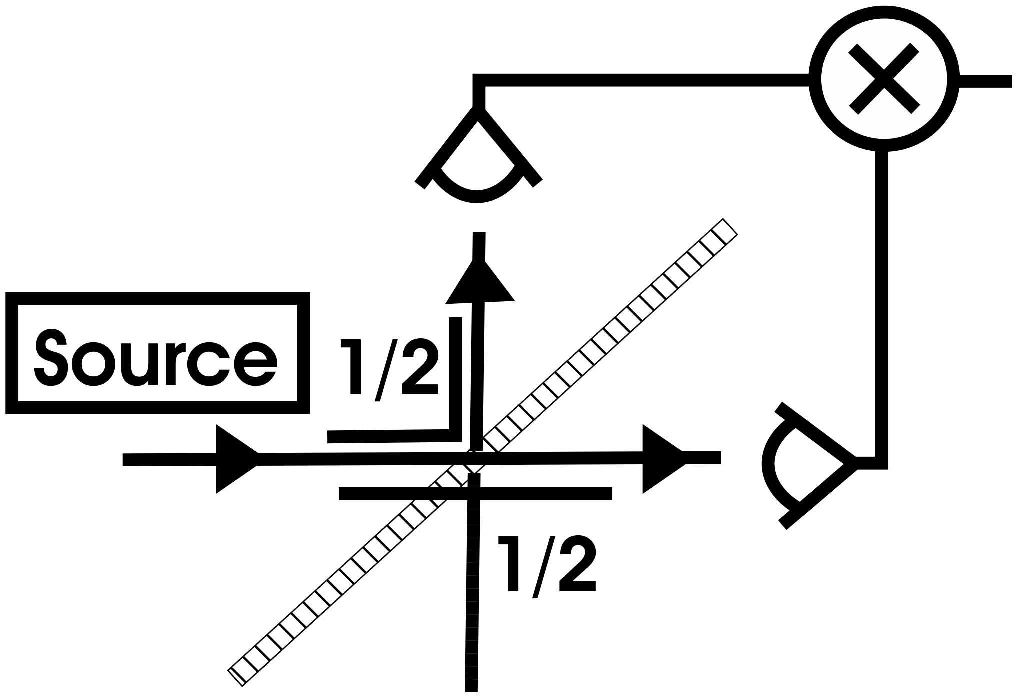


- Random noise: X' and X" (real and imag part) of a signal are statistically independent
- The detection on two orthogonal axes eliminates the amplifier noise.
  This work with a single amplifier!
- The DUT noise is detected

$$S_{ud}(f) = \frac{1}{2}\left[S_\alpha(f) - S_\varphi(f)\right]$$

A. Seeger, H. Stoll, 1/f noise and defects in thin metal films, proc. ICNF p.162-167, Hong Kong 23-26 aug 1999
RF/microwave version: E. Rubiola, V. Giordano, H. Stoll, IEEE Transact. IM 52(1) pp.182-188, feb 2003

Hanbury Brown - Twiss effect

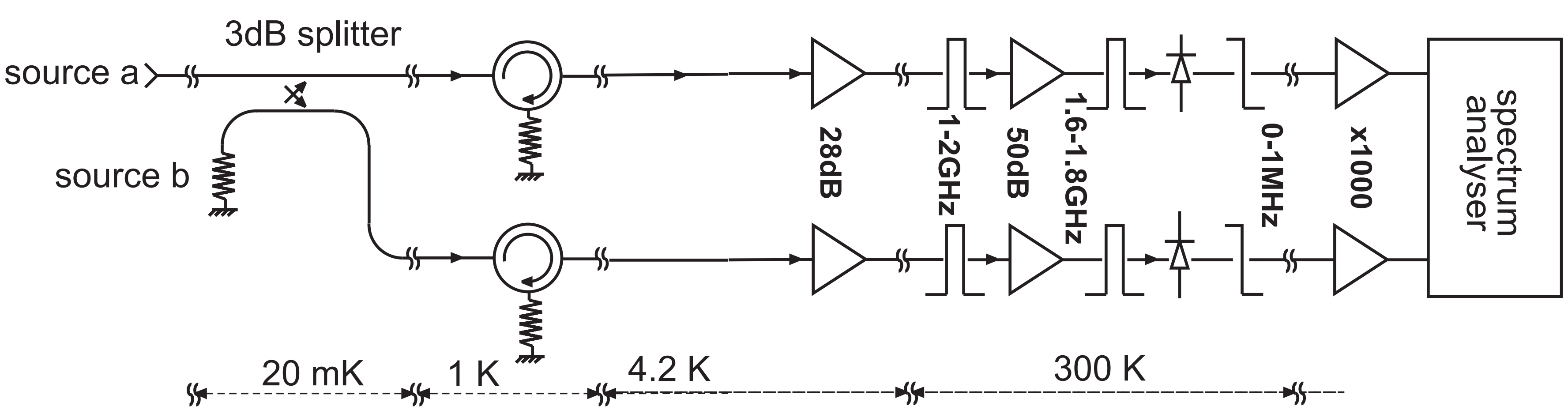


in single-photon regime, anti-correlation shows up

R. Hanbury Brown, R. Q. Twiss, Correlation between photons in two coherent beams of light, Nature 177(27), 1956

Also observed at microwave frequencies

C. Glattli & al. (2004), arXiv:cond-mat/0403584v1 [cond-mat.mes-hall]



$kT = 2.7{\times}10^{-25}$ J, $h\nu = 1.12{\times}10^{-24}$ J, $kT/h\nu = -6.1$ dB

Conclusions

- Correlation enables the rejection of the instrument noise
- In AM noise, RIN, etc., correlation enables the validation of the instrument without a reference low-noise source
- Display quantities
  - $\langle Re\{S_{yx}\}\rangle_m$ is faster and more accurate
  - $|\langle Re\{S_{yx}\}\rangle_m|$ and $|\langle S_{yx}\rangle_m|$ provide easier readout
- Applications in many fields of metrology

The cross spectrum method is magic

Correlated noise sometimes makes magic difficult

home page http://rubiola.org

Part A-1 – The FFT analyzer

Fourier transform

Transform – inverse-transform pair

$$X(\imath f) = \int_{-\infty}^{\infty} x(t)\, e^{-\imath 2\pi f t}\, dt \qquad \leftrightarrow \qquad x(t) = \int_{-\infty}^{\infty} X(\imath f)\, e^{\imath 2\pi f t}\, df$$

Convolution integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Time-convolution theorem

$$x(t) * h(t) \qquad \leftrightarrow \qquad X(\imath f)\, H(\imath f)$$

Frequency-convolution theorem

$$x(t)\, h(t) \qquad \leftrightarrow \qquad X(\imath f) * H(\imath f)$$

Dirac delta function

$$x(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$$

Normalization

Commonly used quantities

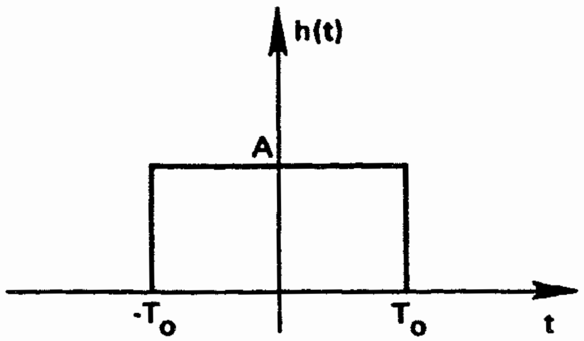
quantity	physical dimension	purpose
$X_T(\imath f)$	V/Hz	Two-sided FT Theoretical issues
$S^I(f) = \frac{2}{T}\lvert X_T(\imath f)\rvert^2, \quad f>0$	V^2/Hz or W/Hz	One-sided PSD Measurement of noise level (power spectral density)
$\frac{1}{T}S^I(f) = \frac{2}{T^2}\lvert X_T(\imath f)\rvert^2, \quad f>0$	V^2 or W	One-sided PS Power measurement of carriers (sinusoidal signals)

Truncated signal

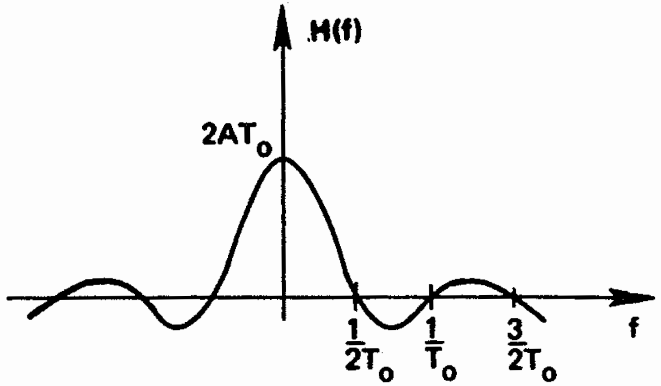
$$X_T(\imath f) = \int_{-T/2}^{T/2} x(t)\, e^{-\imath 2\pi f t}\, dt$$

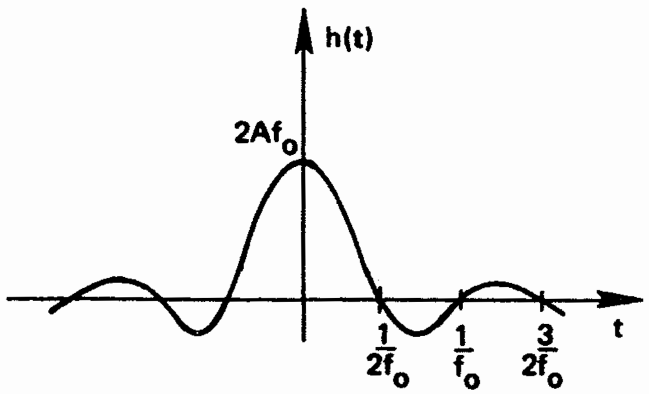
Fourier transform pairs

Time domain | **Frequency domain**

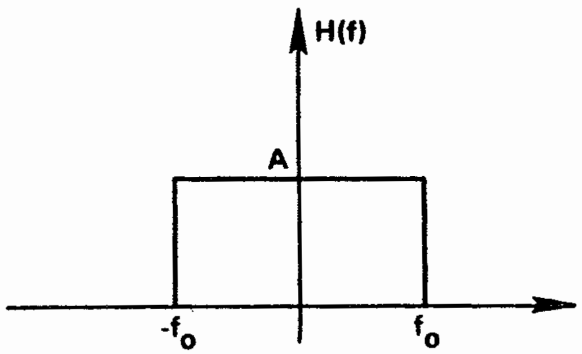


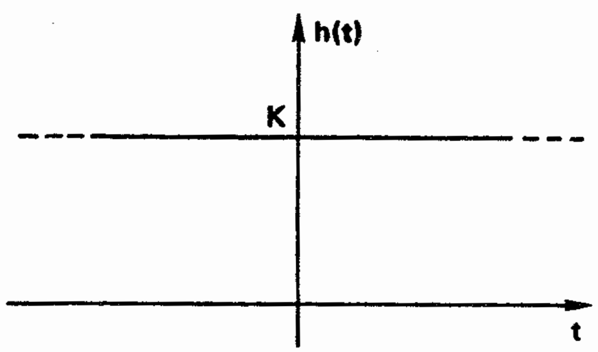
$$h(t) = A \quad |t| < T_0$$
$$= \frac{A}{2} \quad |t| = T_0 \quad \Leftrightarrow \quad H(f) = 2AT_0 \frac{\sin(2\pi T_0 f)}{2\pi T_0 f}$$
$$= 0 \quad |t| > T_0$$



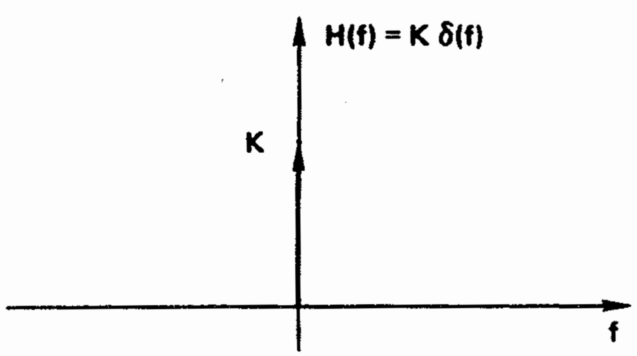


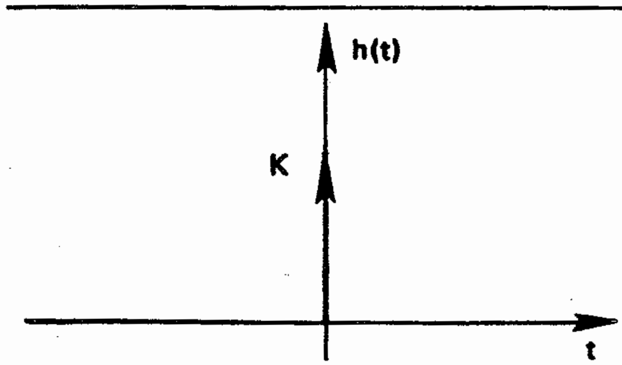
$$h(t) = 2Af_0 \frac{\sin(2\pi f_0 t)}{2\pi f_0 t} \quad \Leftrightarrow \quad H(f) = A \quad |f| < f_0$$
$$= \frac{A}{2} \quad |f| = f_0$$
$$= 0 \quad |f| > f_0$$



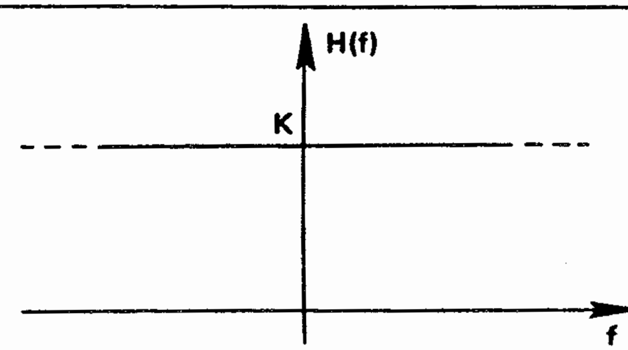


$$h(t) = K \quad \Leftrightarrow \quad H(f) = K\delta(f)$$





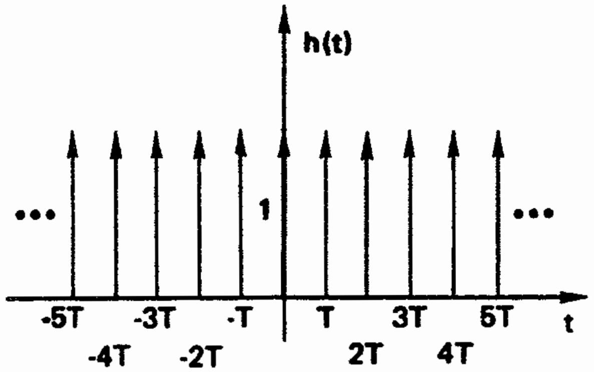
$$h(t) = K\delta(t) \quad \Leftrightarrow \quad H(f) = K$$



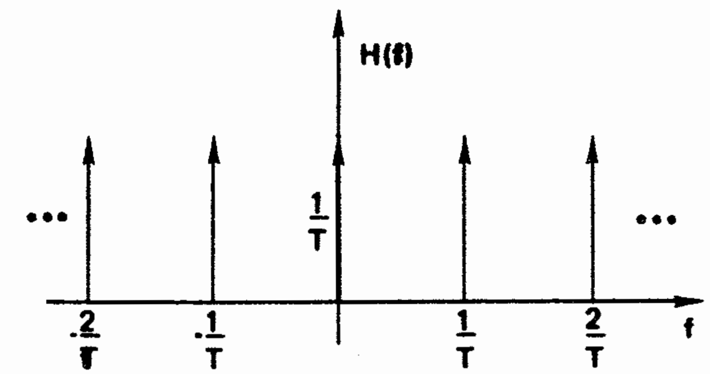
E. Oran Brigham, The fast Fourier Transform, Prentice Hall, 1988

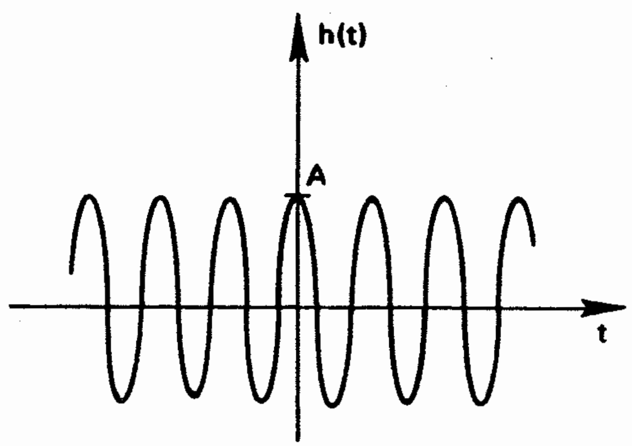
Fourier transform pairs

Time domain | **Frequency domain**

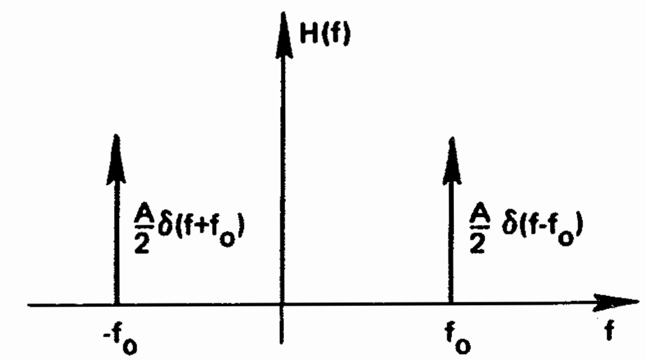


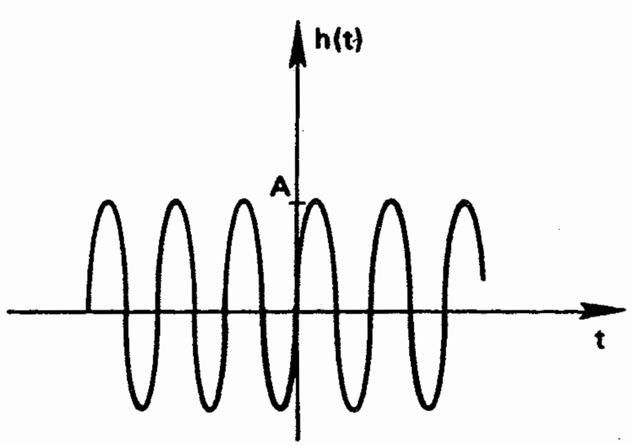
$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Longleftrightarrow H(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$



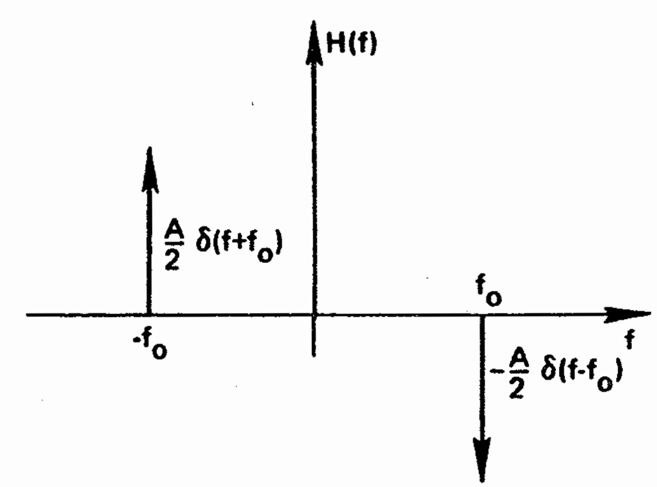


$$h(t) = A\cos(2\pi f_0 t) \Longleftrightarrow H(f) = \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0)$$





$$h(t) = A\sin(2\pi f_0 t) \Longleftrightarrow H(f) = -j\frac{A}{2}\delta(f - f_0) + j\frac{A}{2}\delta(f + f_0)$$

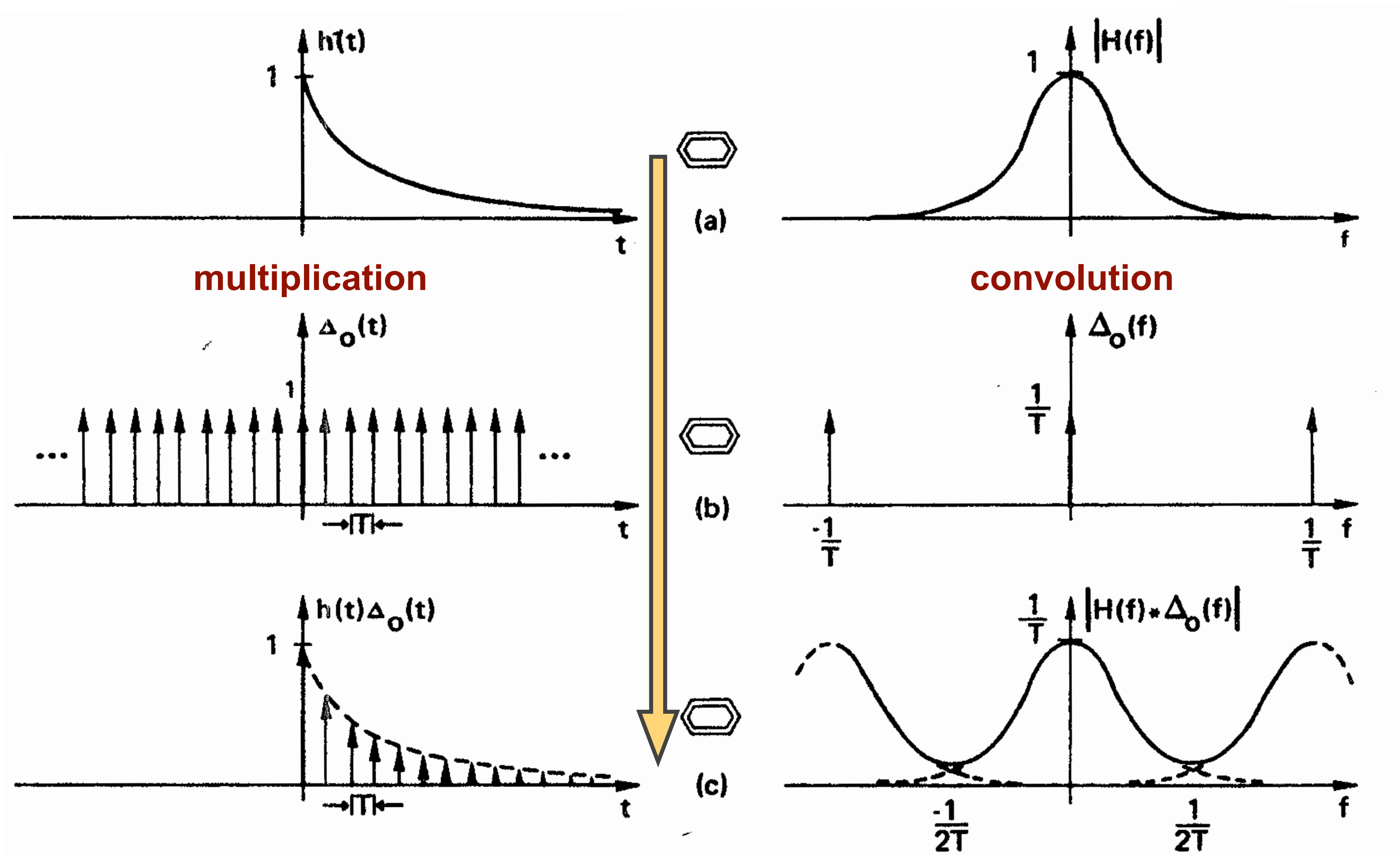


E. Oran Brigham, The fast Fourier Transform, Prentice Hall, 1988

Sampling and aliasing

Time domain

Frequency domain



Input signal

(Time-domain) sampling

Sampled signal (and aliasing)

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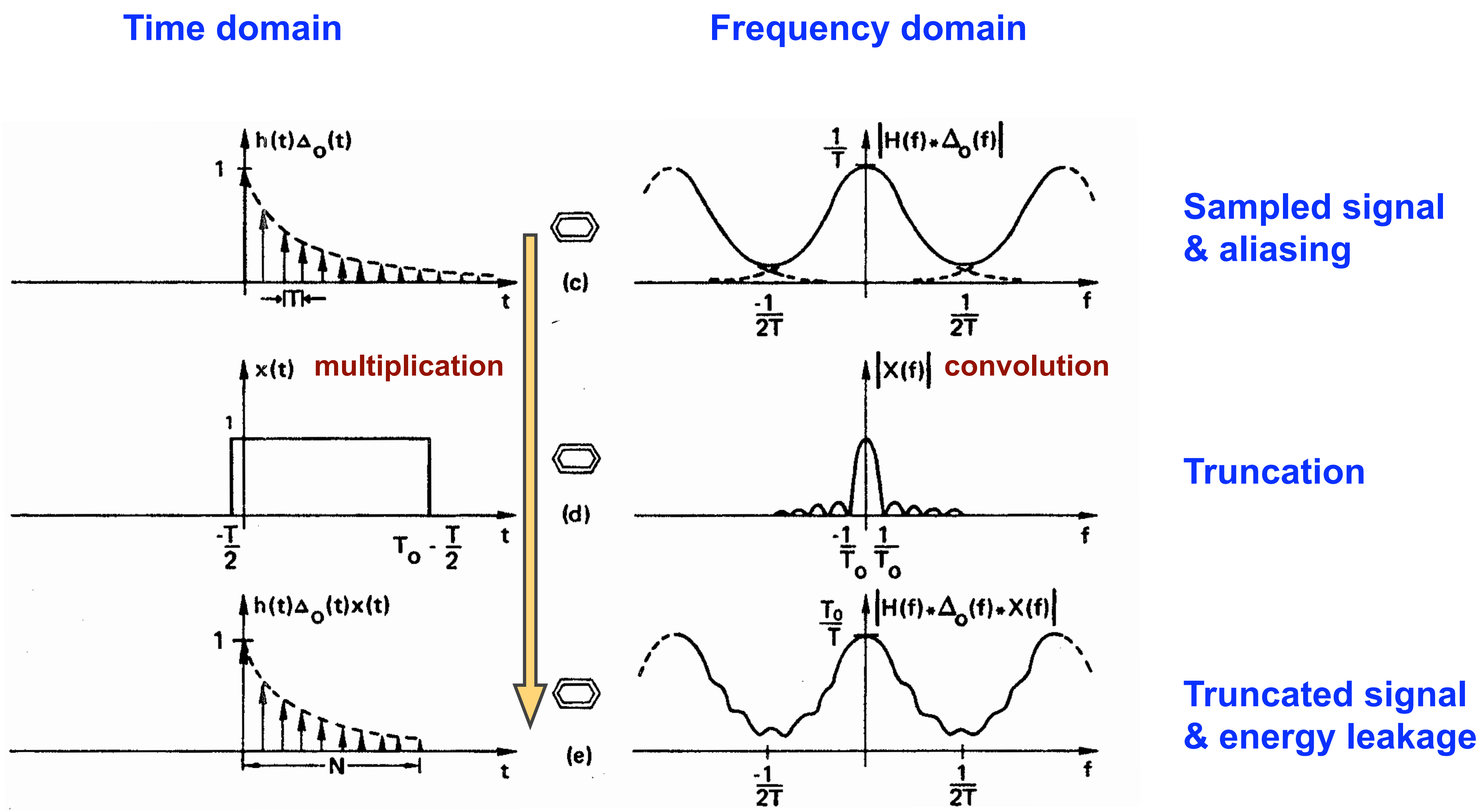
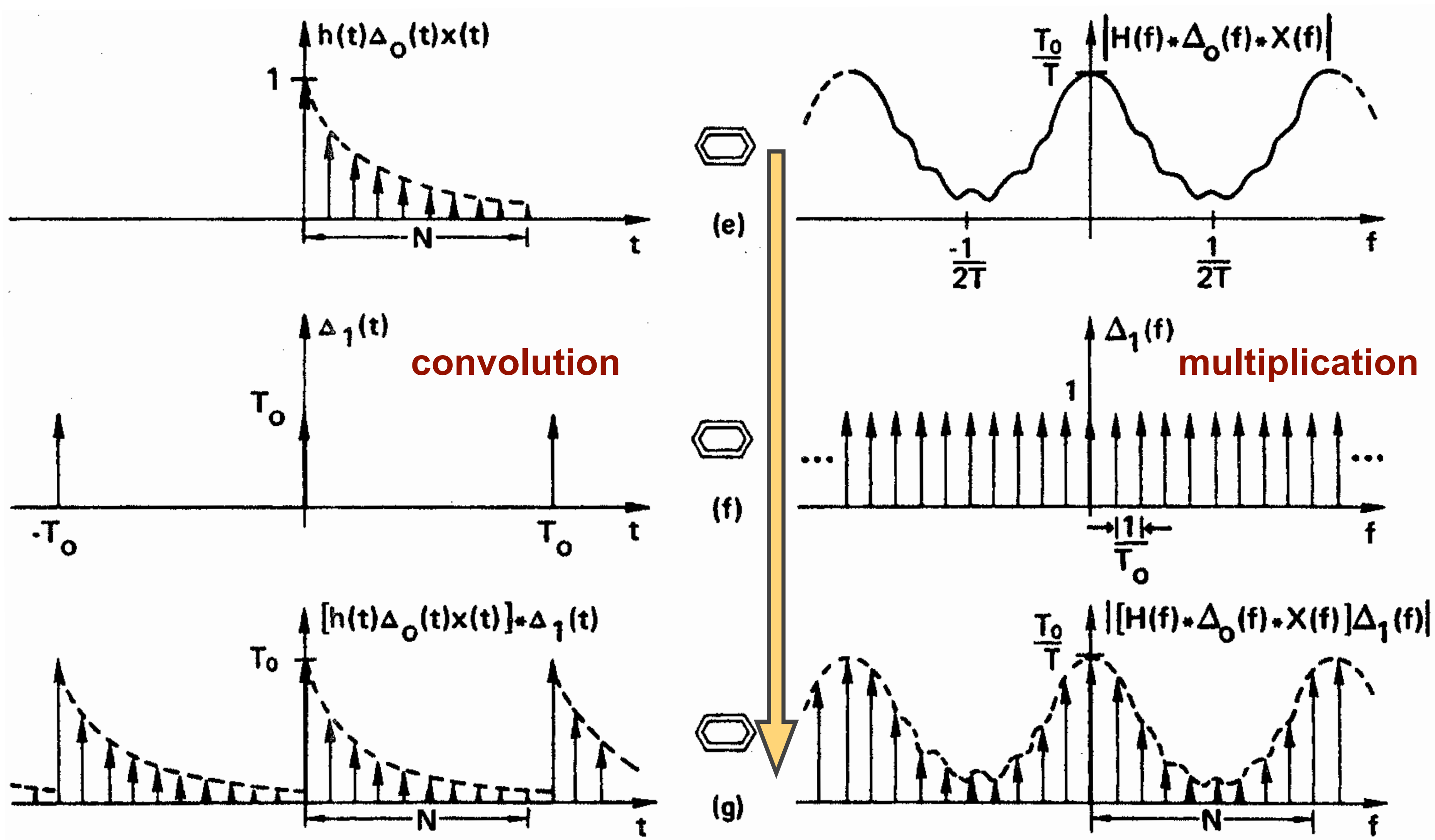
Truncation and energy leakage



E. Oran Brigham, The fast Fourier Transform, Prentice Hall, 1988

Fitting the Fourier transform into a computer memory

Time domain **Frequency domain**



Truncated signal

Frequency-domain sampling

Final DFT (Time-domain aliasing)

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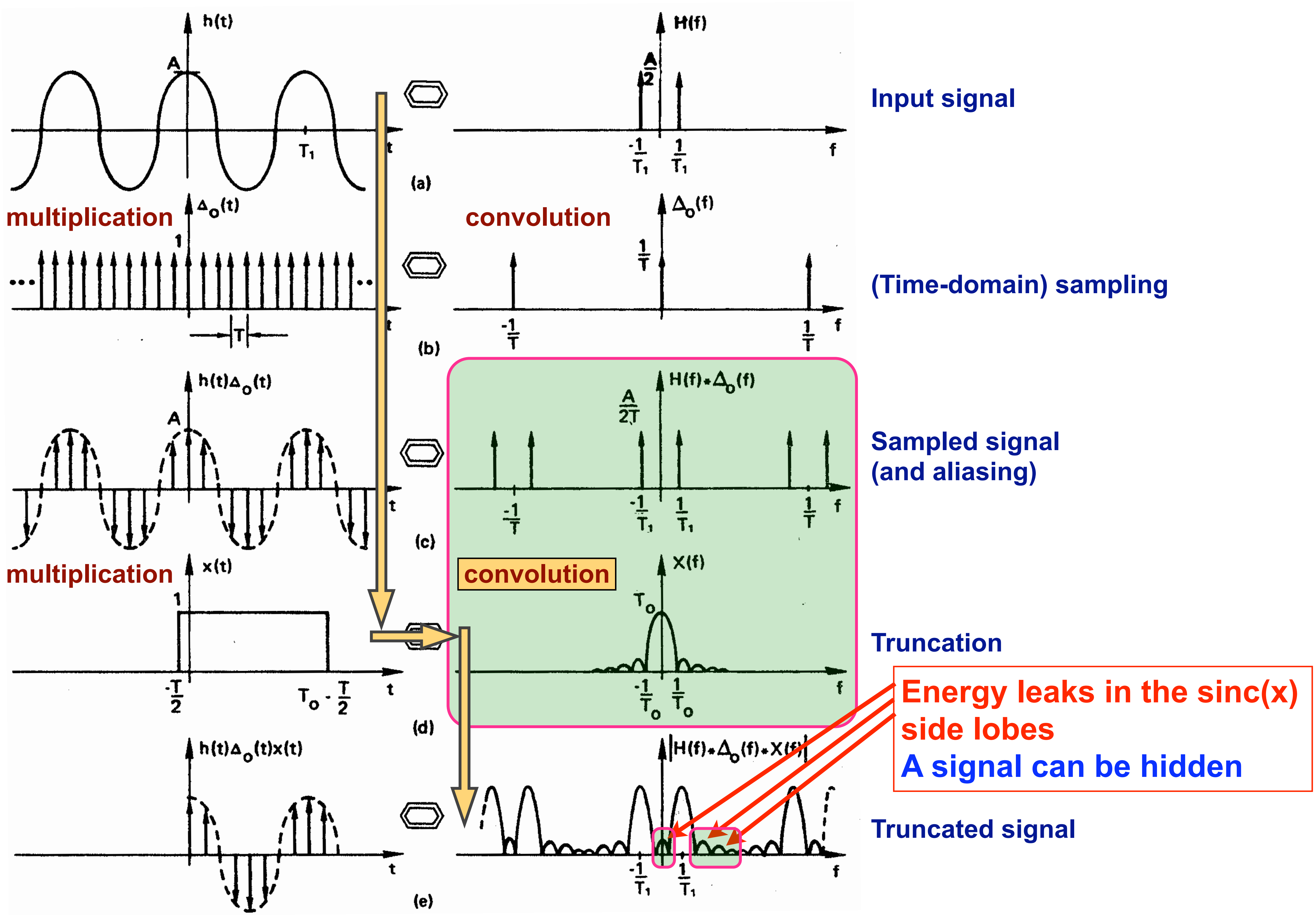
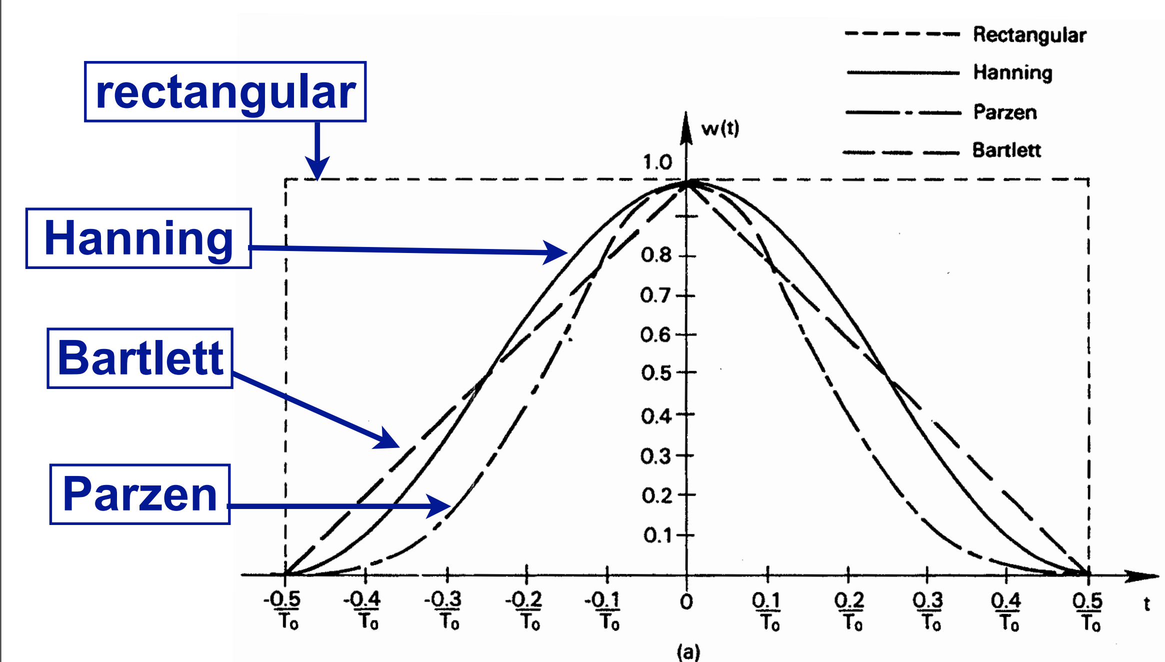
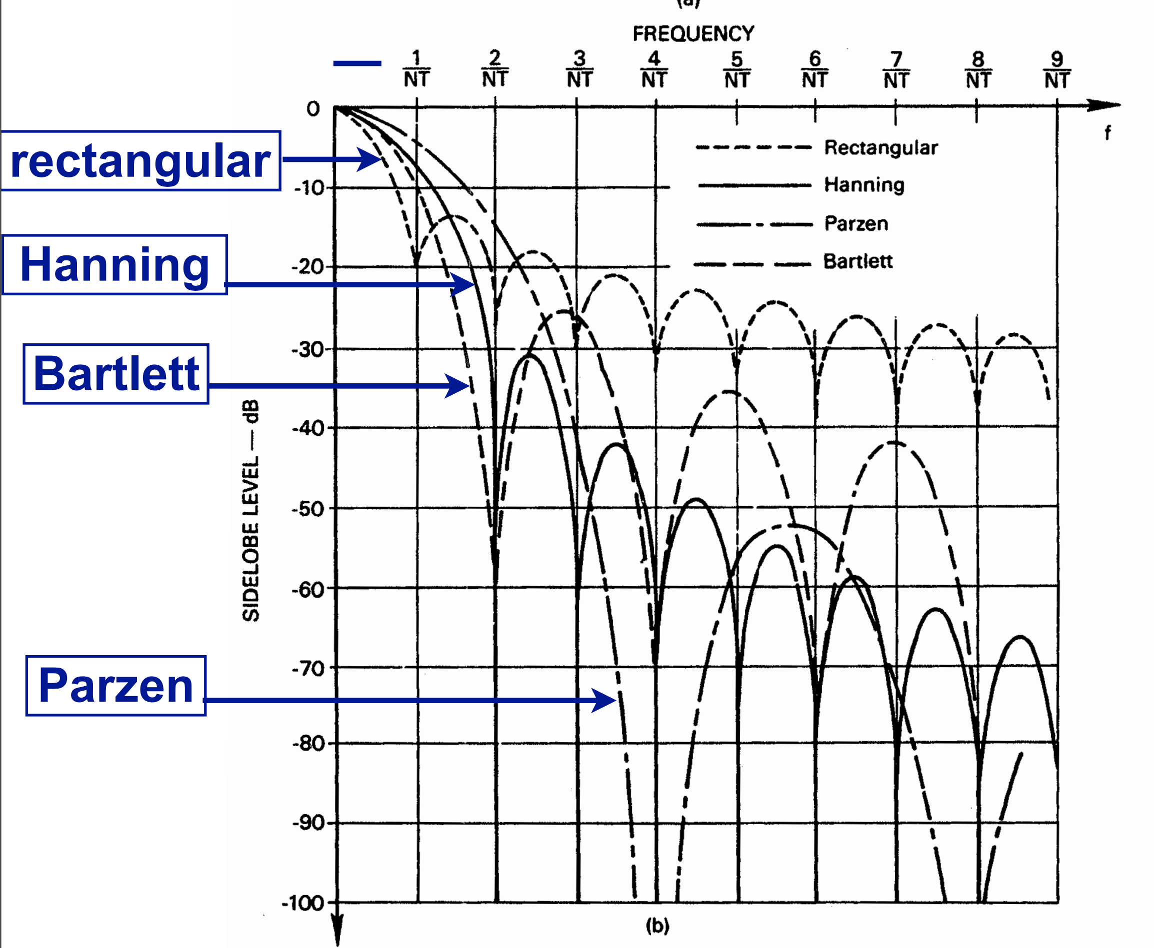
Windowing – the problem

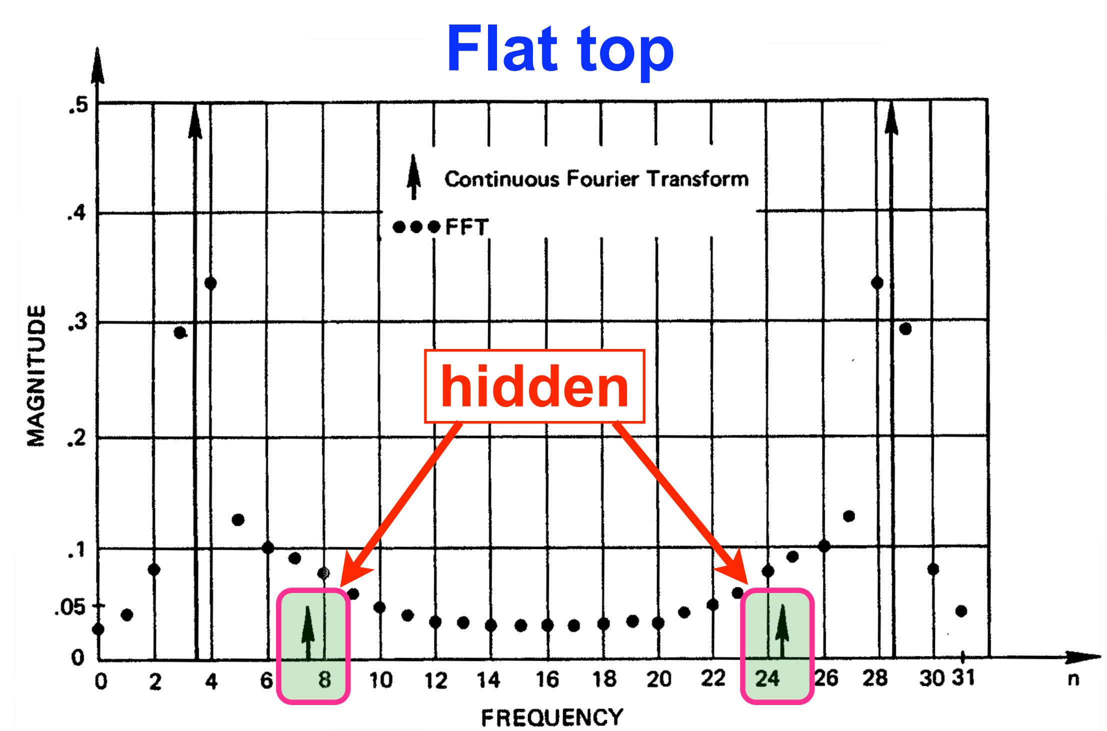


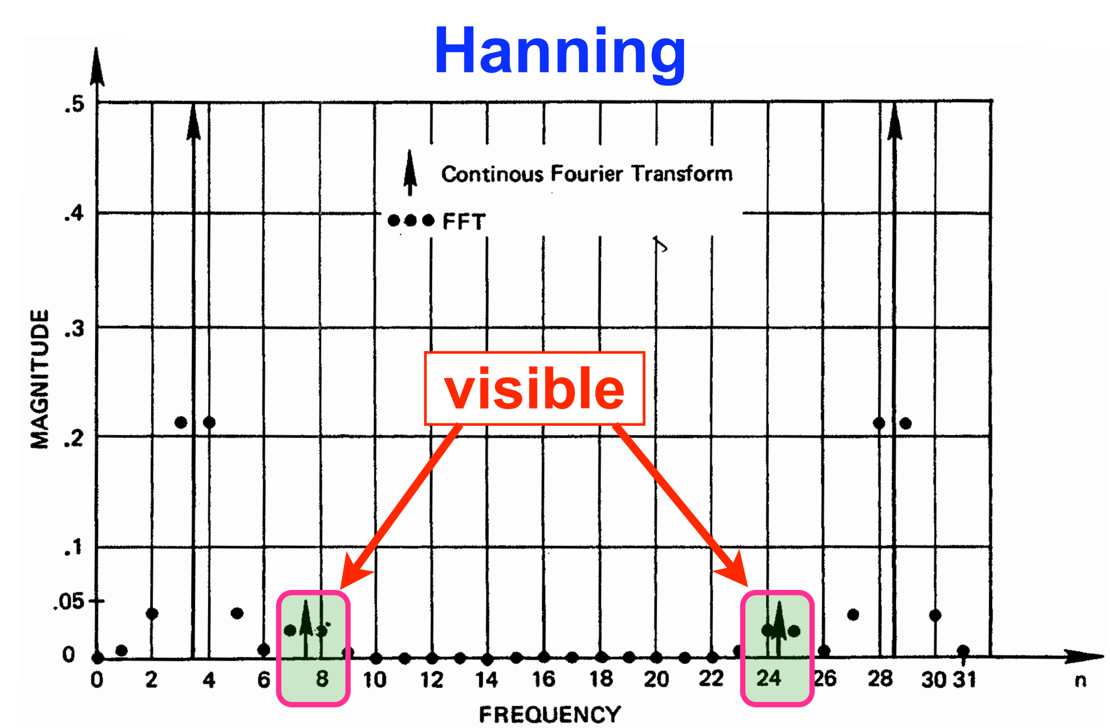
E. Oran Brigham, The fast Fourier Transform, Prentice Hall, 1988

Windowing – solution









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Window functions

Weighting Function Nomenclature	Time Domain	Frequency Domain	Highest Side-Lobe Level (db)	3-dB Bandwidth	Asymptotic Rolloff (dB/Octave)
Rectangular	$w_R(t) = 1 \quad \lvert t \rvert \le \frac{T_0}{2}$ $= 0 \quad \lvert t \rvert > \frac{T_0}{2}$	$W_R(f) = \frac{T_0 \sin(\pi f T_0)}{\pi f T_0}$	−13	$\frac{0.85}{T_0}$	6
Bartlett (triangle)	$w_B(t) = \left[1 - \frac{2\lvert t \rvert}{T_0}\right] \quad \lvert t \rvert < \frac{T_0}{2}$ $= 0 \quad \lvert t \rvert > \frac{T_0}{2}$	$W_B(f) = \frac{T_0}{2}\left[\frac{\sin\left(\frac{\pi}{2} f T_0\right)}{\frac{\pi}{2} f T_0}\right]^2$	−26	$\frac{1.25}{T_0}$	12
Hanning (cosine)	$w_H(t) = \cos^2\left(\frac{\pi t}{T_0}\right)$ $= \frac{1}{2}\left[1 + \cos\left(\frac{2\pi t}{T_0}\right)\right] \quad \lvert t \rvert \le \frac{T_0}{2}$ $= 0 \quad \lvert t \rvert > \frac{T_0}{2}$	$W_H(f) = \frac{T_0}{2} \frac{\sin(\pi f T_0)}{\pi f T_0 [1 - (f T_0)^2]}$	−32	$\frac{1.4}{T_0}$	18
Parzen	$w_P(t) = 1 - 24\left(\frac{t}{T_0}\right)^2 + 48\left\lvert \frac{t}{T_0} \right\rvert^3 \quad \lvert t \rvert < \frac{T_0}{4}$ $= 2\left[1 - \frac{2\lvert t \rvert}{T_0}\right]^3 \quad \frac{T_0}{4} < \lvert t \rvert < \frac{T_0}{2}$ $= 0 \quad \lvert t \rvert \ge \frac{T_0}{2}$	$W_P(f) = \frac{3T_0}{8}\left[\frac{\sin(\pi f T_0/4)}{\pi f T_0/4}\right]^4$	−52	$\frac{1.82}{T_0}$	24

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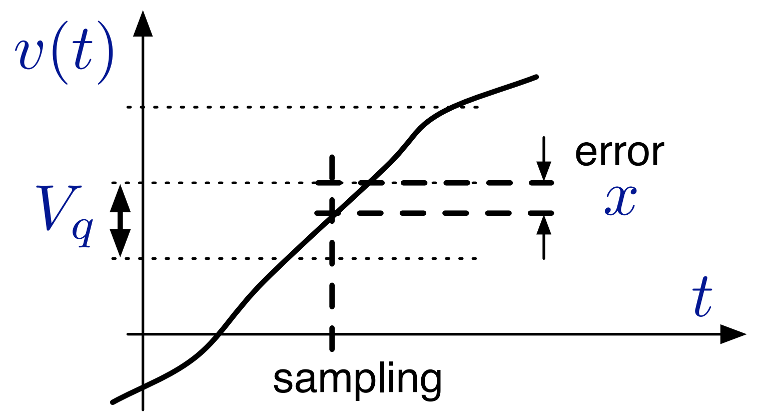
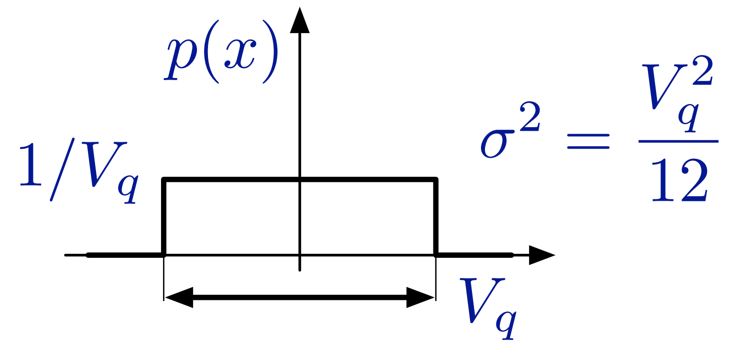
Spectrum of the quantization noise

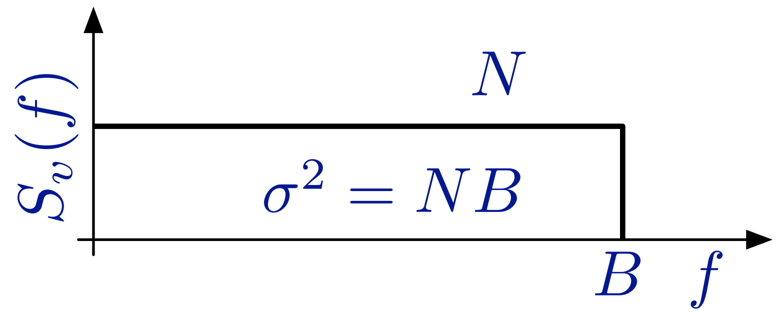


The analog-to-digital converter introduces a quantization error x, $-V_q/2 \le x \le +V_q/2$



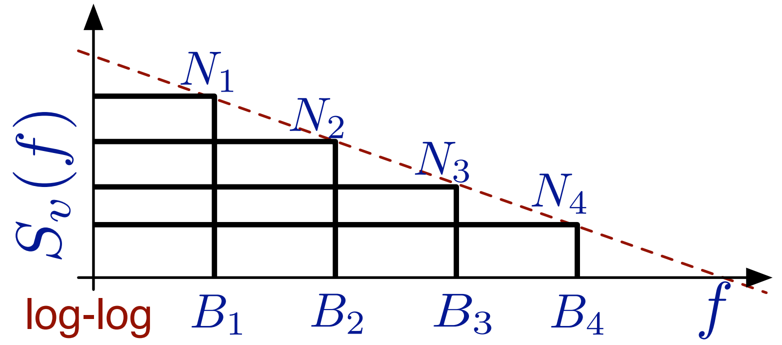
Ergodicity suggests that the quantization noise can be calculated statistically

$$\sigma^2 = \frac{V_q^2}{12}$$



The Parseval theorem states that energy and power can be evaluated by integrating the spectrum

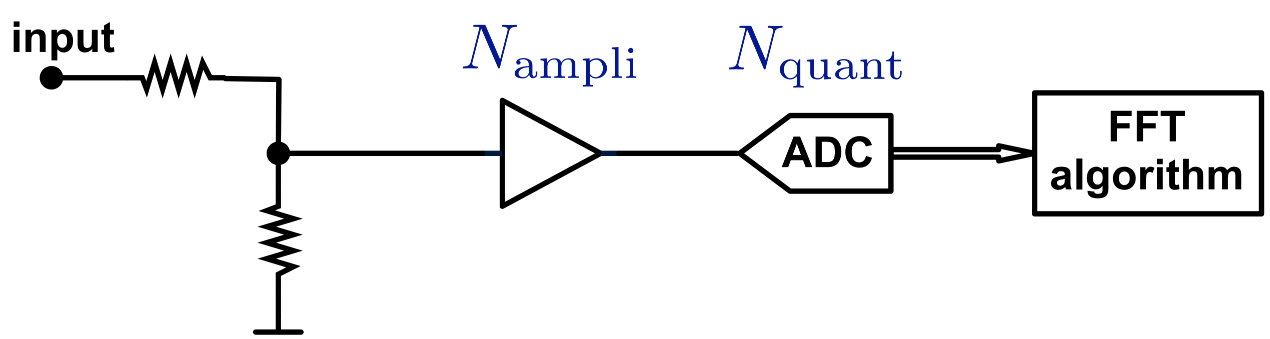
$$NB = \frac{V_q^2}{12}$$

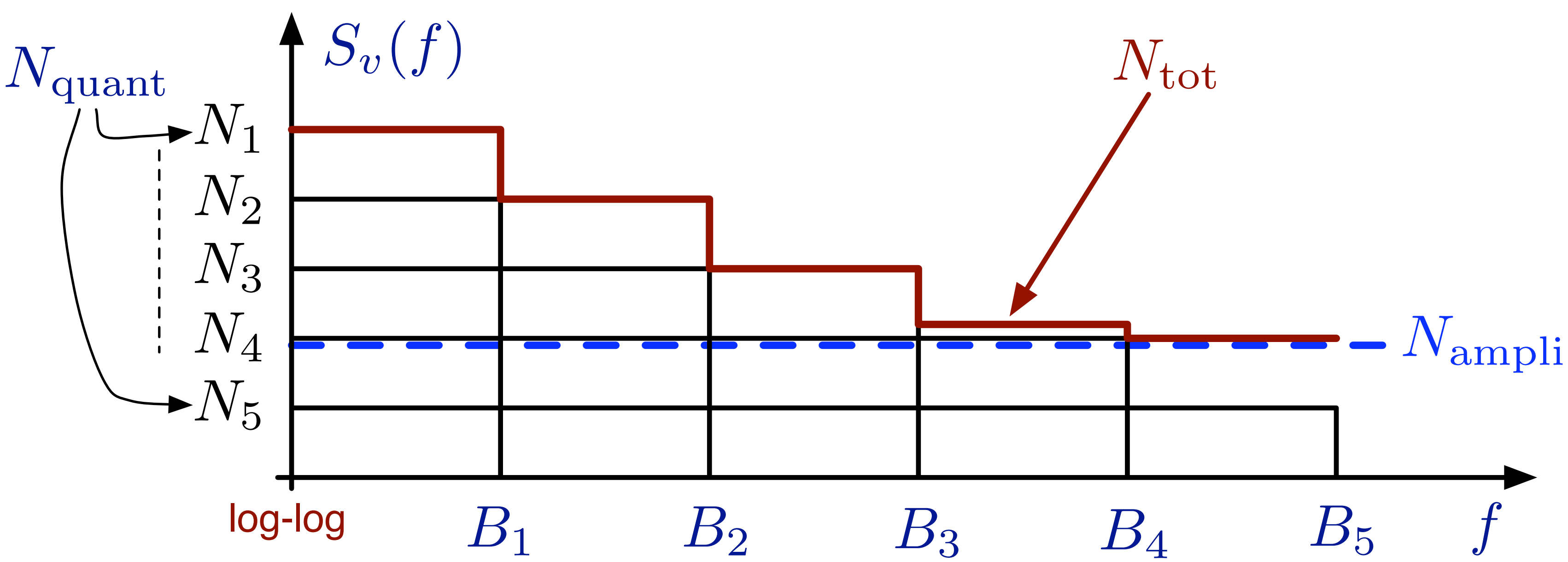


Changing B in geometric progression (decades) yields naturally 1/B (flicker) noise

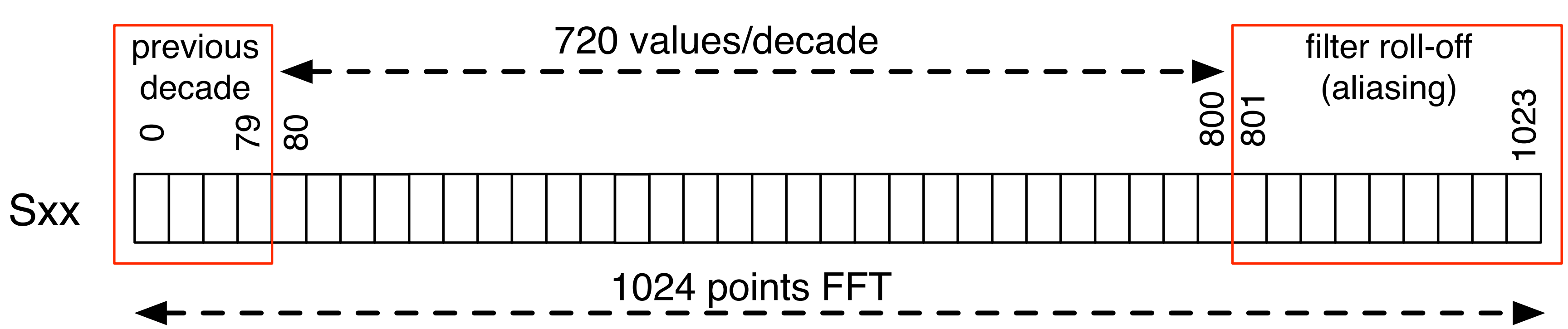
$$N = \frac{V_q^2}{12B}$$

Noise of the real FFT analyzer





The quantization noise scales with the frequency span, the front-end noise is constant

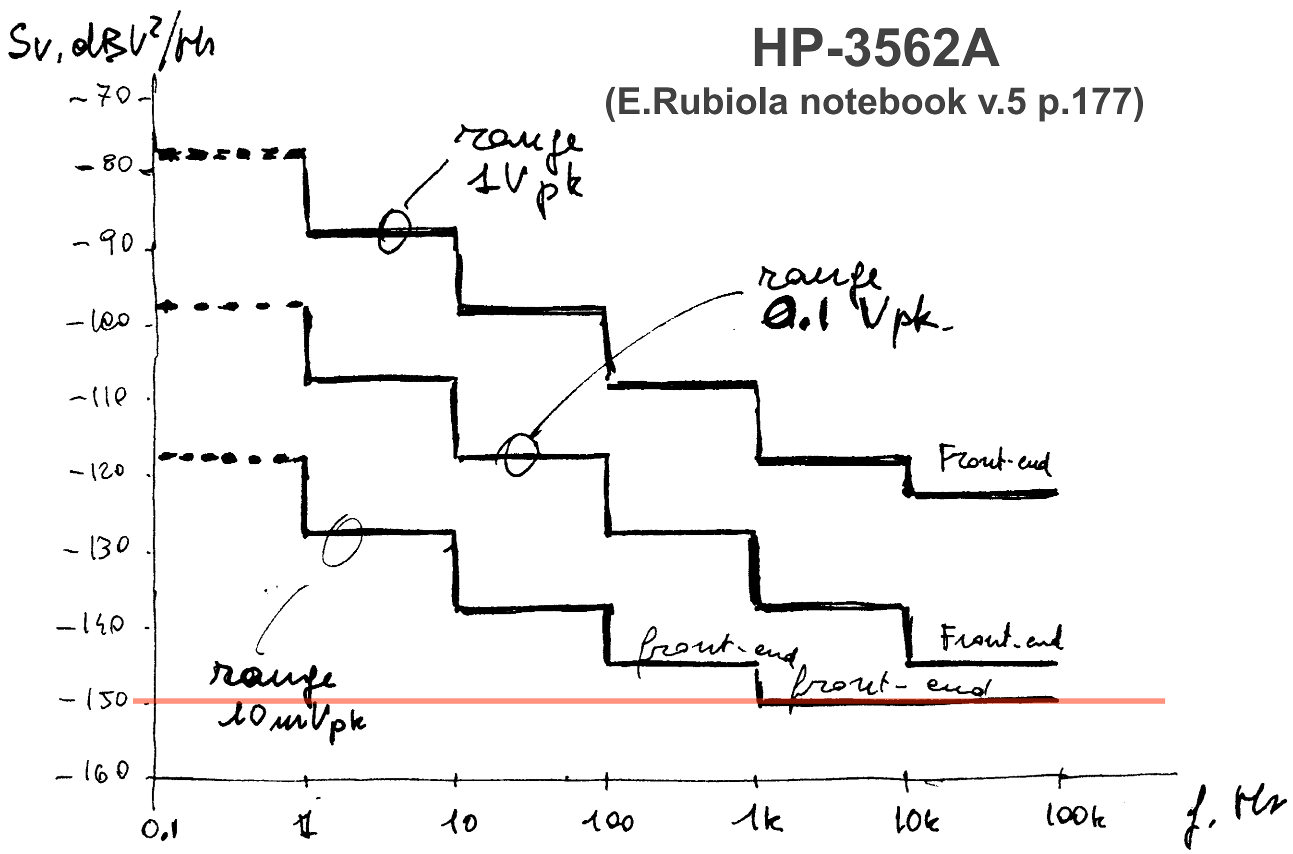


The energy is equally spread in the full FFT bandwidth, including the upper region not displayed because of aliasing

Example of FFT analyzer noise

Experimental observation

HP-3562A
(E.Rubiola notebook v.5 p.177)



Theoretical evaluation

DAC 12 bit resolution, including sign

range 10 mV_{peak}
V_{fsr} = 20 mV (±10 mV)

resolution
$V_q = V_{fsr} / 2^{12}$
= 4.88 μV

total noise
$\sigma^2 = (4.88\ \mu V)^2 / 12$
$= 2{\times}10^{-12}\ V^2$ (–117 dB)

quantization noise PSD
$S_v = \sigma^2 / B$
= –117 dBV^2/Hz with B = 1 Hz (etc.)

Front-end noise, evaluated from the plot

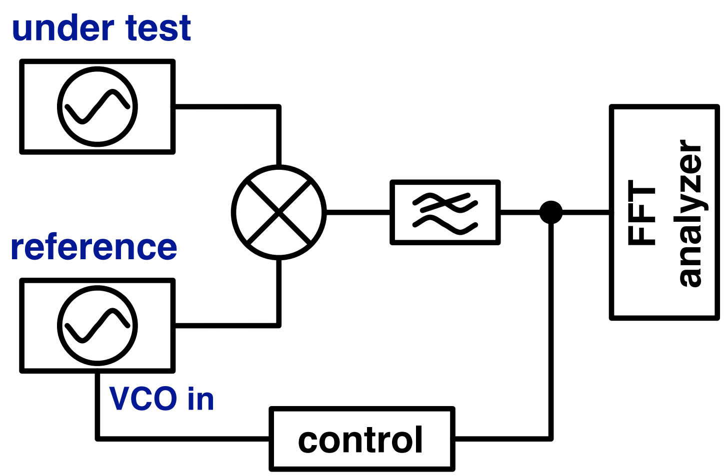
$S_v = 2{\times}10^{-15}\ V^2$ (–150 dB), at 10–100 kHz
or 45 $nV/Hz^{1/2}$

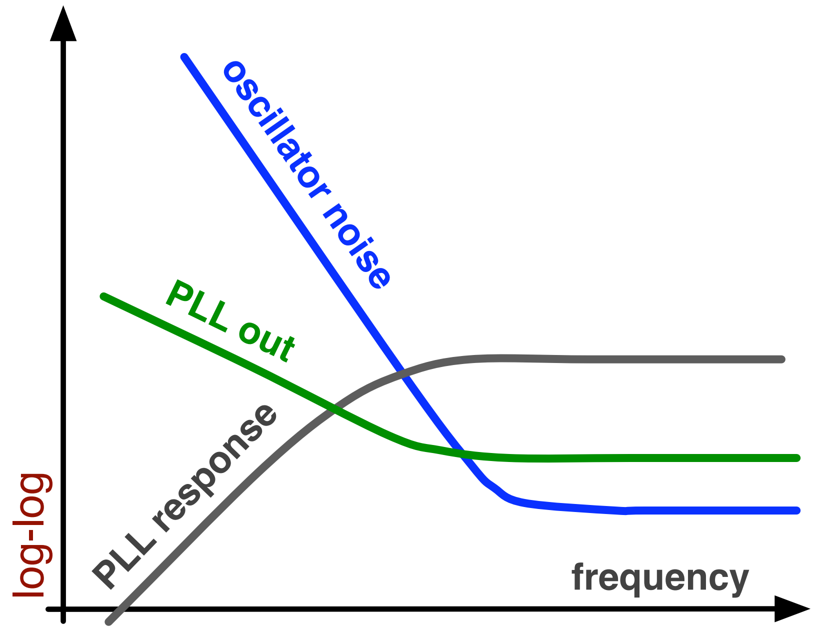
use Sv = 4kTR
R = 125 kΩ
or R = 100 kΩ and F = 1 dB (noise figure)

Oscillator noise measurement

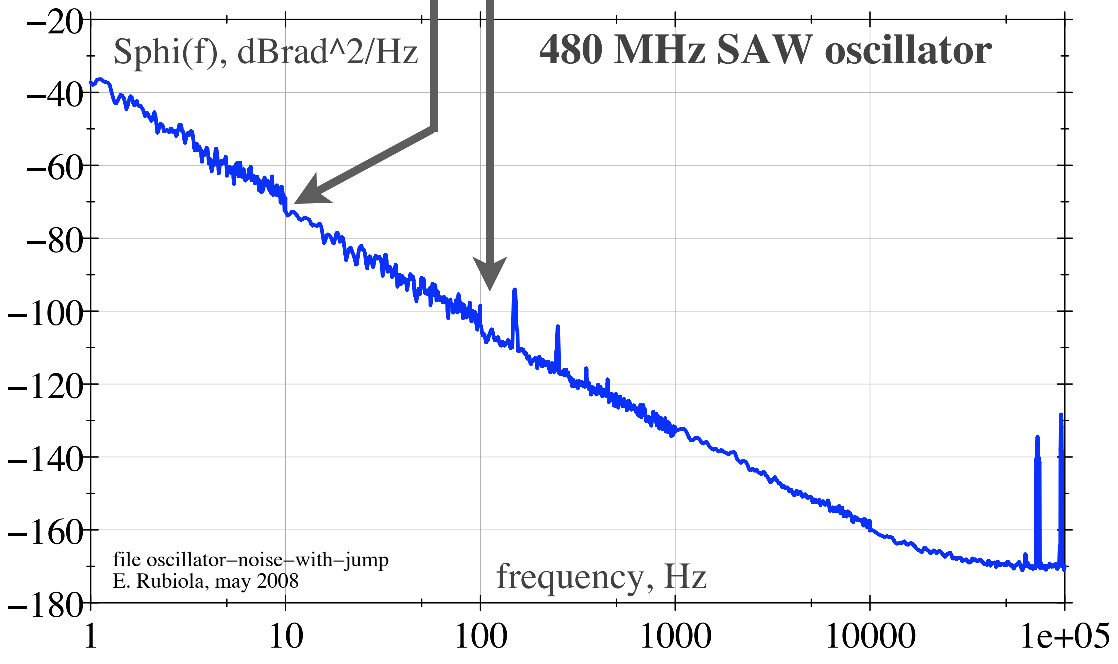
A tight loop is preferred because:

– reduces the required dynamic range
– overrides (parasitic) injection locking





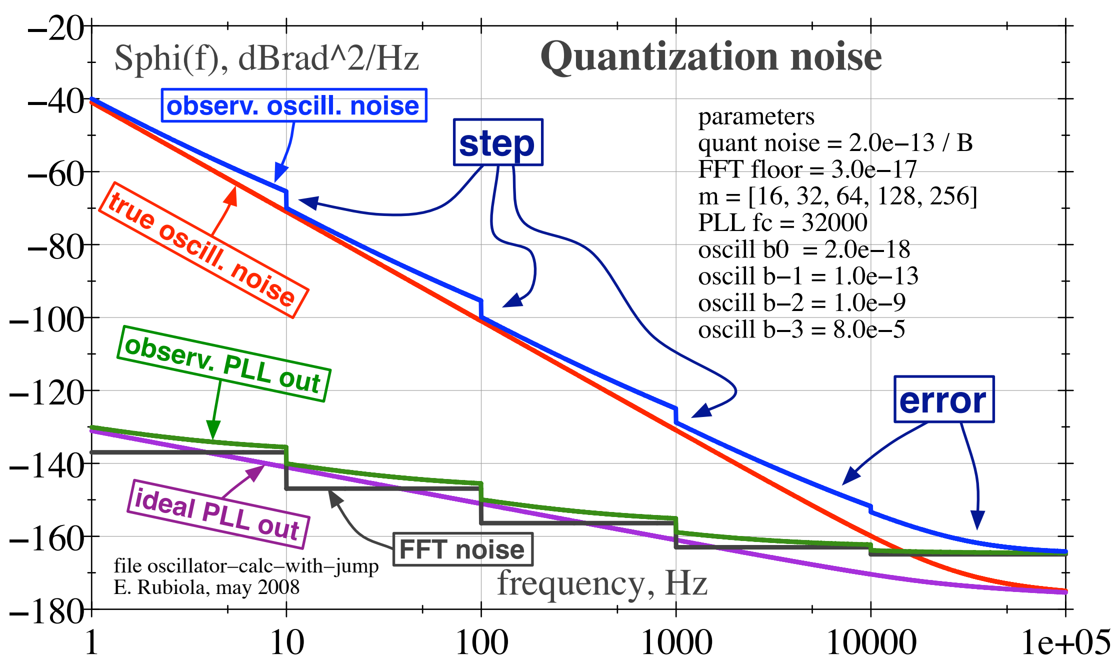
Steps are sometimes observed, due to the FFT quantization noise



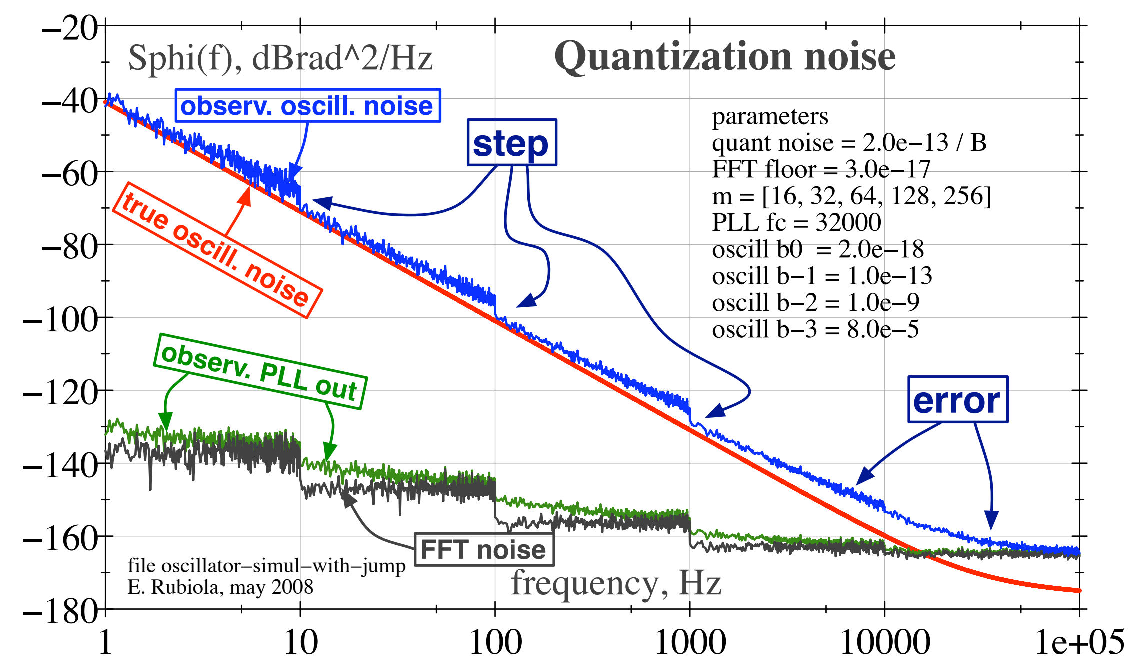
FFT noise in oscillator measurements

Explanation: the steps occurring at the transition between decades are due the quantization noise, when the resolution is insufficient

calculated



simulated



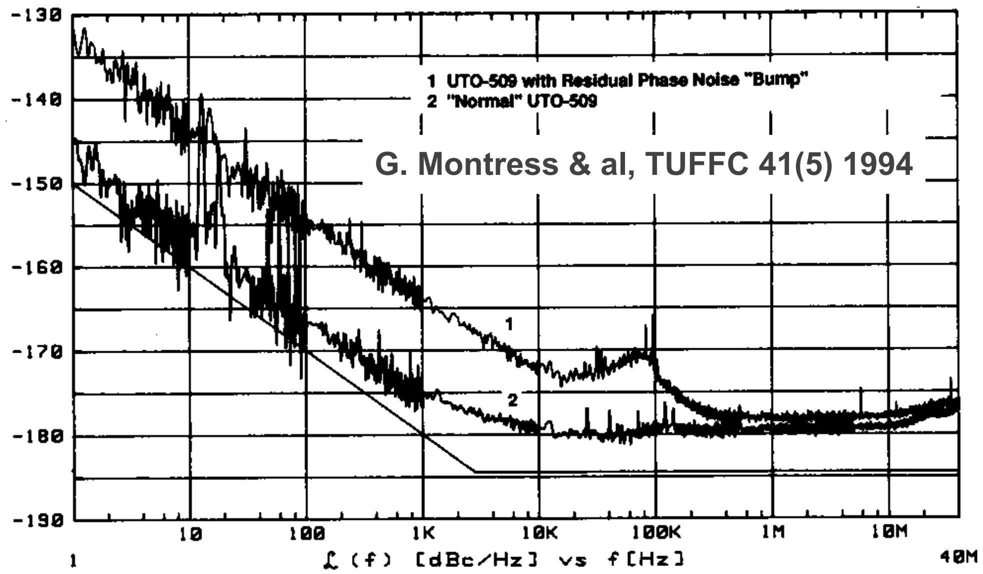
The steps are due to the FFT quantization noise

The problem shows up when the dynamic range is insufficient, often in the presence of large stray signals

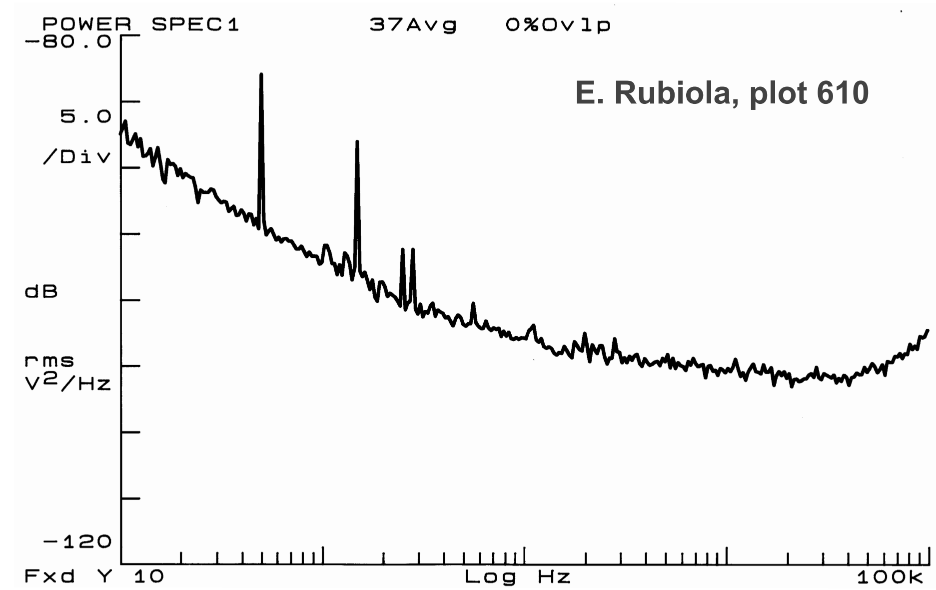
Systematic errors are also possible at high Fourier frequencies

Linear vs. logarithmic resolution

Linear resolution

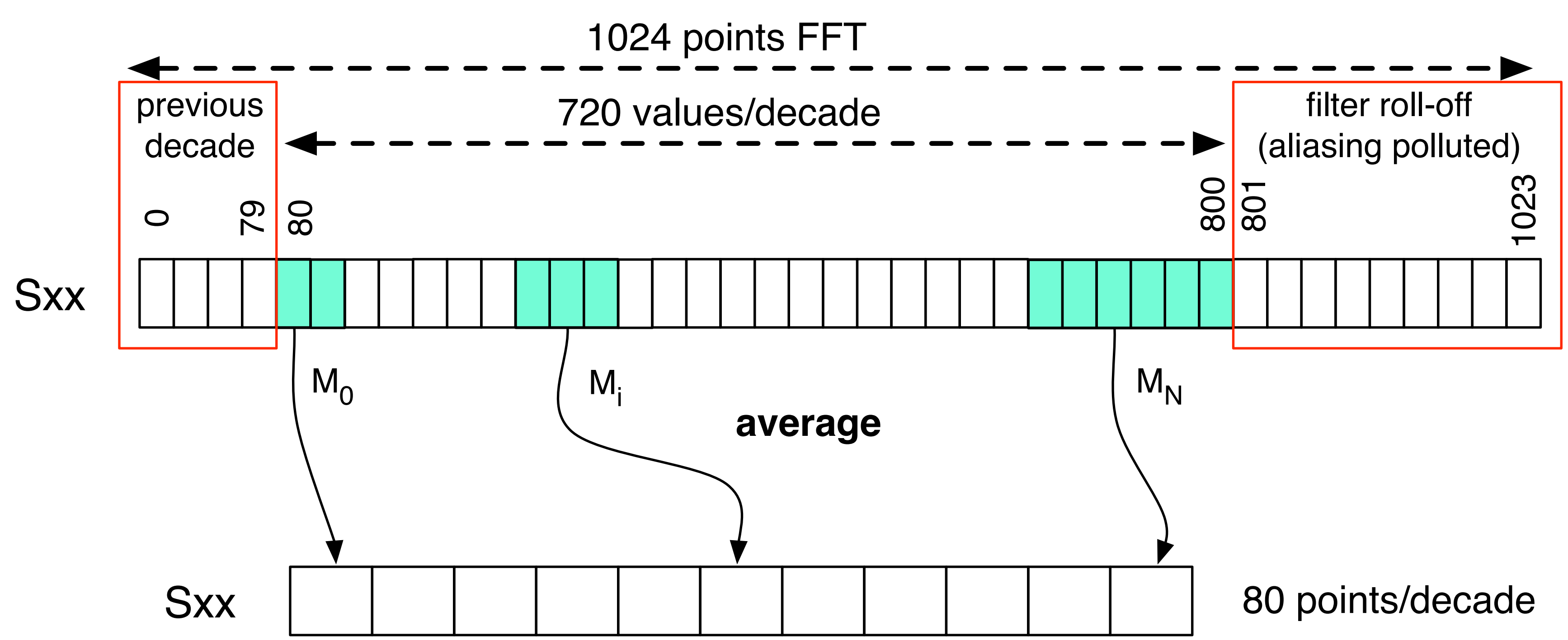


Logarithmic resolution (80 pt/dec)



Combining M independent values, the confidence interval is reduced by sqrt(M), (5 dB left-right in one decade)

A weighted average is also possible



Part A-2 – Statistics

Properties of white zero-mean gaussian noise

$$x(t) \iff X(\imath f) = X'(\imath f) + \imath X''(\imath f)$$

1. $x(t) \iff X(\imath f)$ are gaussian
2. $X(\imath f_1)$ and $X(\imath f_2)$ are uncorrelated
   $var\{X(\imath f_1)\} = var\{X(\imath f_2)\}$
3. X' and X'' are uncorrelated
   $var\{X'\} = var\{X''\} = var\{X\}/2$
4. $Y = X_1 + X_2$ is gaussian
   $var\{Y\} = var\{X_1\} + var\{X_2\}$
5. $Y = X_1 \times X_2$ is gaussian
   $var\{Y\} = var\{X_1\}\, var\{X_2\}$

Properties of flicker noise

$$x(t) \iff X(\imath f) = X'(\imath f) + \imath X''(\imath f)$$

1. $x(t) \iff X(\imath f)$, there is no a-priori relationship between the distribution of $x(t)$ and $X(\imath f)$ (theorem).
   Central limit theorem => $X(\imath f)$ can be gaussian

2. $X(\imath f_1)$ and $X(\imath f_2)$ are correlated.
   correlation decays rapidly when $f_1 \neq\neq f_2$
   $\mathrm{var}\{X(\imath f_1)\} \neq \mathrm{var}\{X(\imath f_2)\}$

3. X' and X'' can be correlated
   $\mathrm{var}\{X'\} \neq \mathrm{var}\{X''\} \neq \mathrm{var}\{X\}/2$

4. $Y = X_1 + X_2$, with zero-mean X_1, X_2,
   $\mathrm{var}\{Y\} = \mathrm{var}\{X_1\} + \mathrm{var}\{X_2\}$

5. If X_1 and X_2 are zero-mean gaussian r.v.
   then $Y = X_1 \times X_2$ is zero-mean gaussian
   and $\mathrm{var}\{Y\} = \mathrm{var}\{X_1\}\ \mathrm{var}\{X_2\}$

One-sided gaussian distribution

x is normal distributed with zero mean and variance σ²

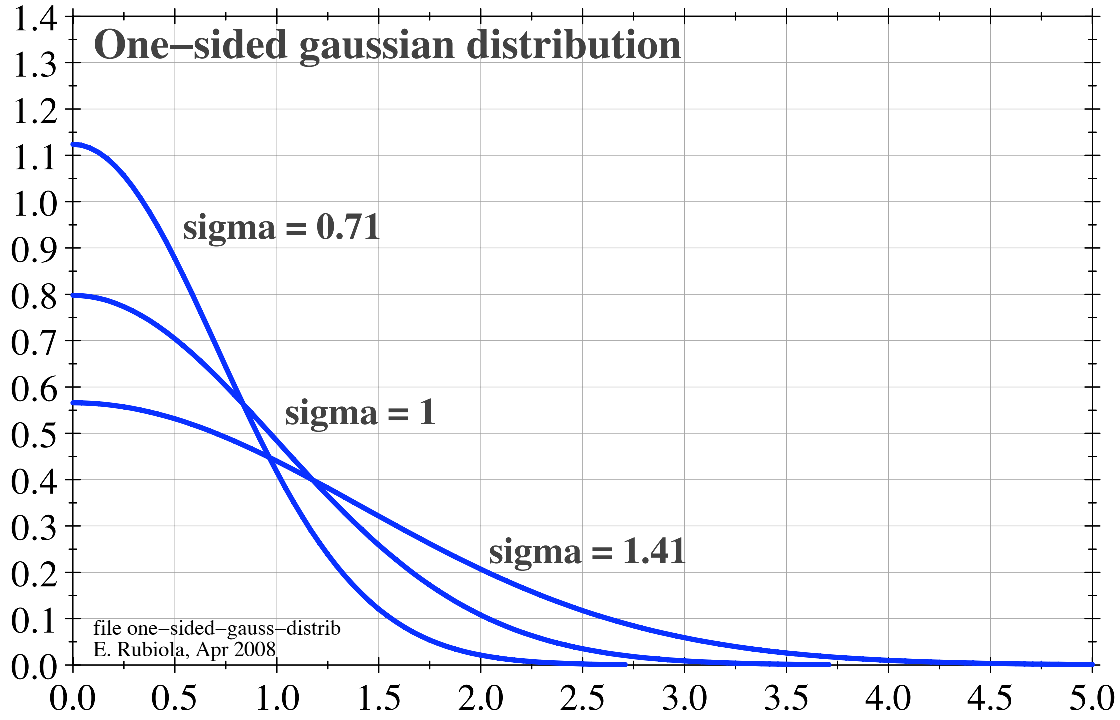
$$y = |x|$$

$$f(x) = 2\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad y \ge 0$$

$$\mathbb{E}\{f(x)\} = \sqrt{\frac{2}{\pi}}\,\sigma$$

$$\mathbb{E}\{f^2(x)\} = \sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \left(1 - \frac{2}{\pi}\right)\sigma^2$$



one-sided gaussian distribution with $\sigma^2 = 1/2$	
quantity with $\sigma^2 = 1/2$	value [10 log( ), dB]
average $= \sqrt{\frac{1}{\pi}}$	0.564 [−2.49]
deviation $= \sqrt{\frac{1}{2} - \frac{1}{\pi}}$	0.426 [−3.70]
$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{\pi}{2} - 1}$	0.756 [−1.22]
$\frac{\text{avg} + \text{dev}}{\text{avg}} = 1 + \sqrt{\frac{1}{2} - \frac{1}{\pi}}$	1.756 [+2.44]
$\frac{\text{avg} - \text{dev}}{\text{avg}} = 1 - \sqrt{\frac{1}{2} - \frac{1}{\pi}}$	0.244 [−6.12]
$\frac{\text{avg} + \text{dev}}{\text{avg} - \text{dev}} = \frac{1 + \sqrt{1/2 - 1/\pi}}{1 - \sqrt{1/2 - 1/\pi}}$	7.18 [8.56]

Chi-square distribution

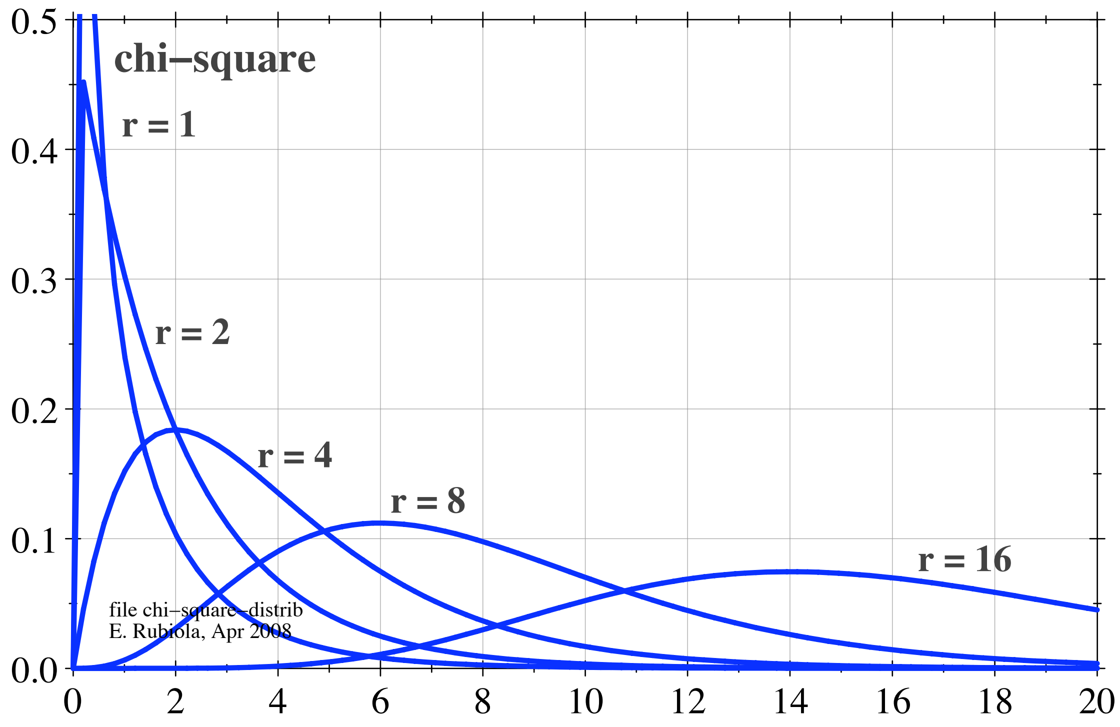
x_i are normal distributed with zero mean and equal variance σ^2

$$\chi^2 = \sum_{i=1}^{r} x_i^2$$

is χ^2 distributed with r degrees of freedom

Notice that the sum of χ^2 is a χ^2 distribution

$$\chi^2 = \sum_{j=1}^{m} \chi_j^2 \;, \qquad r = \sum_{j=1}^{m} r_j$$



$$f(x) = \frac{x^{\frac{r}{2}-1}\, e^{-\frac{x^2}{2}}}{\Gamma\left(\frac{1}{2}r\right) 2^{\frac{r}{2}}} \qquad x \geq 0$$

$$\mathbb{E}\{f(x)\} = \sigma^2 r$$

$$\mathbb{E}\{[f(x)]^2\} = \sigma^4 r(r+2)$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = 2\sigma^4 r$$

$$z! = \Gamma(z+1), \qquad z \in \mathbb{N}$$

Averaging m chi-square distributions

averaging m variables $|X|^2$, complex X=X'+ıX", yields a χ^2 distribution with r = 2m

$$\frac{1}{m}\,\chi^2 = \frac{1}{m}\sum_{j=1}^{m}(X'_j)^2 + (X''_j)^2$$

$$\mathbb{E}\left\{\frac{1}{m}\,f(x)\right\} = \frac{\sigma^2 r}{m} = 2\sigma^2$$

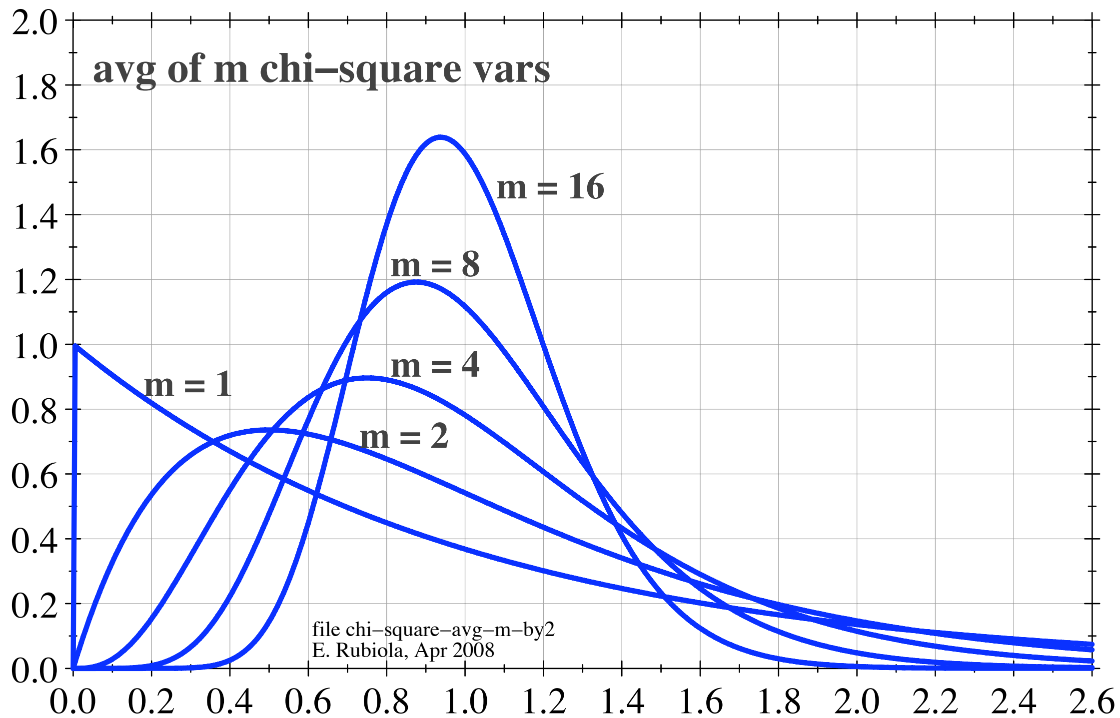
$$\mathbb{E}\left\{\left|\frac{1}{m}\,f(x) - \mathbb{E}\left\{\frac{1}{m}\,f(x)\right\}\right|^2\right\} = \frac{2\sigma^4 r}{m^2} = \frac{4\sigma^4}{m}$$

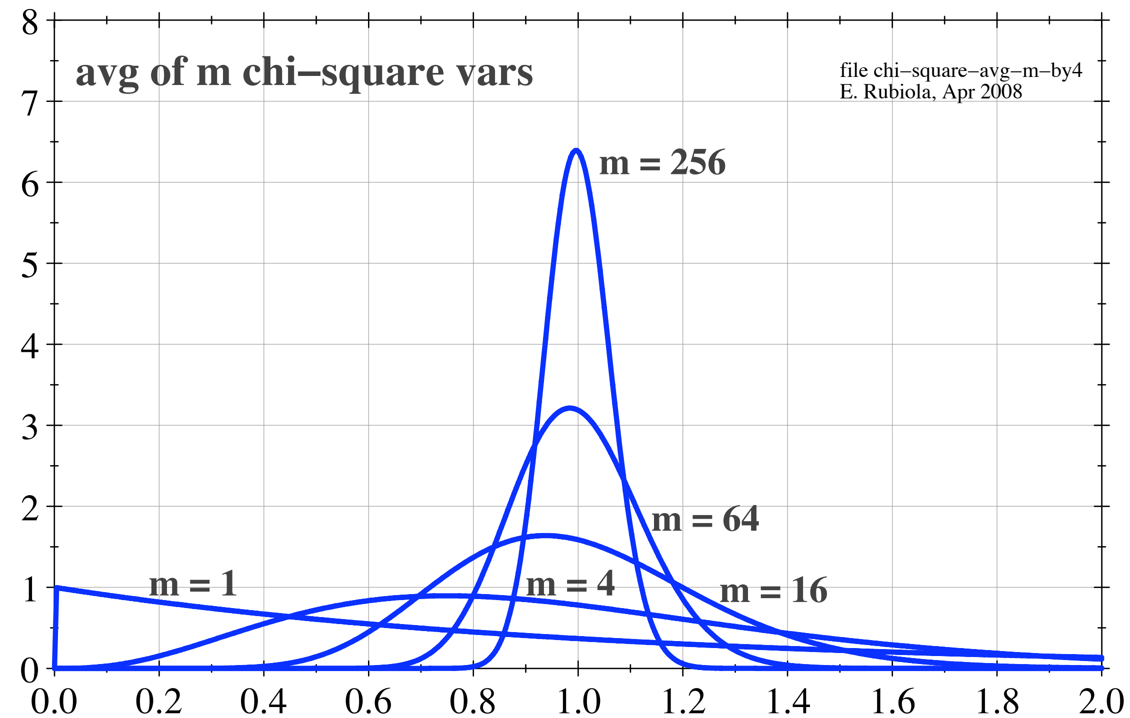
$$\frac{\text{dev}}{\text{avg}} = \frac{1}{\sqrt{m}}$$

relevant case: $\sigma^2 = 1/2$

$\text{avg} = 1$

$\text{dev} = \dfrac{1}{\sqrt{m}}$

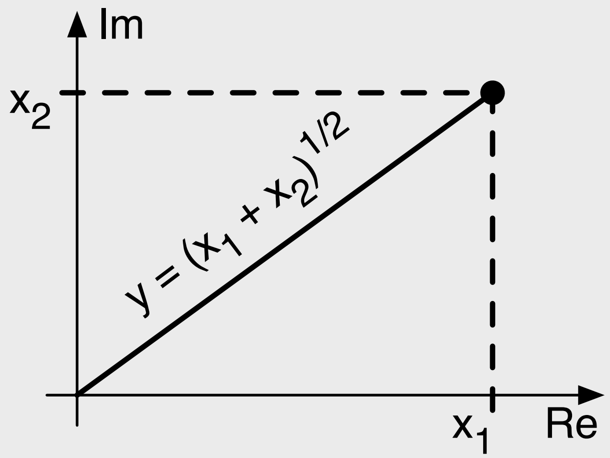




Rayleigh distribution

x_1 and x_2 are normal distributed with zero mean and equal variance σ^2

$$x = \sqrt{x_1^2 + x_2^2}$$



x is Rayleigh-distributed

✔

$$f(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad x \geq 0$$

$$\mathbb{E}\{f(x)\} = \sqrt{\frac{\pi}{2}}\,\sigma$$

$$\mathbb{E}\{f^2(x)\} = 2\sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \frac{4-\pi}{2}\sigma^2$$

✔

Rayleigh distribution with $\sigma^2 = 1/2$	
quantity with $\sigma^2 = 1/2$	value [10 log( ), dB]
average $= \sqrt{\frac{\pi}{4}}$	0.886 [−0.525]
deviation $= \sqrt{1 - \frac{\pi}{4}}$	0.463 [−3.34]
$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{4}{\pi} - 1}$	0.523 [−2.82]
$\frac{\text{avg} + \text{dev}}{\text{avg}} = 1 + \sqrt{\frac{4}{\pi} - 1}$	1.523 [+1.83]
$\frac{\text{avg} - \text{dev}}{\text{avg}} = 1 - \sqrt{\frac{4}{\pi} - 1}$	0.477 [−3.21]
$\frac{\text{avg} + \text{dev}}{\text{avg} - \text{dev}} = \frac{1 + \sqrt{4/\pi - 1}}{1 - \sqrt{4/\pi - 1}}$	3.19 [5.04]