Scientific Instruments

— and —

Phase Noise and Frequency Stability in Oscillators

Lectures for PhD Students and Young Scientists

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Part 1: General

Part 2: Phase noise and oscillators

Part 3: The International System of Units SI

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Lecture 1

Scientific Instruments & Oscillators

Lectures for PhD Students and Young Scientists

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Contents

- Quantum noise
- Thermal noise
- Shot noise

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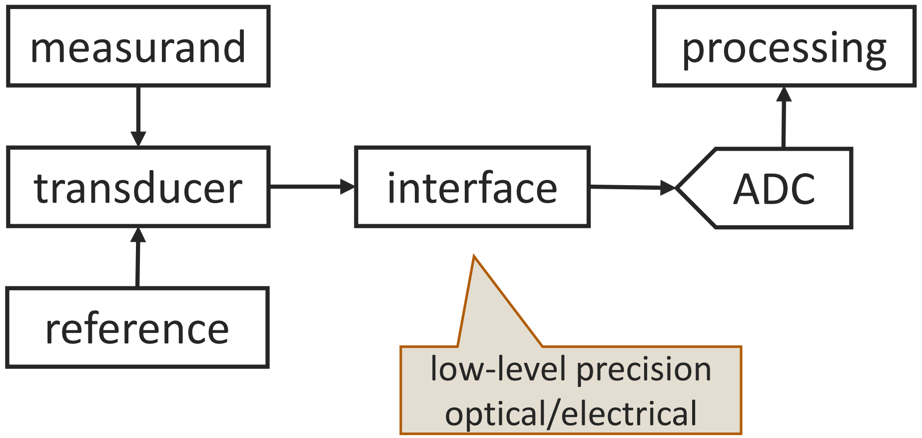
home page http://rubiola.org

Instruments

Featured reading: P. Horowitz, W. Hill, *The Art of Electronics*, 3rd ed, Cambridge 2015

Instruments

basic scheme



measurand

transducer

reference

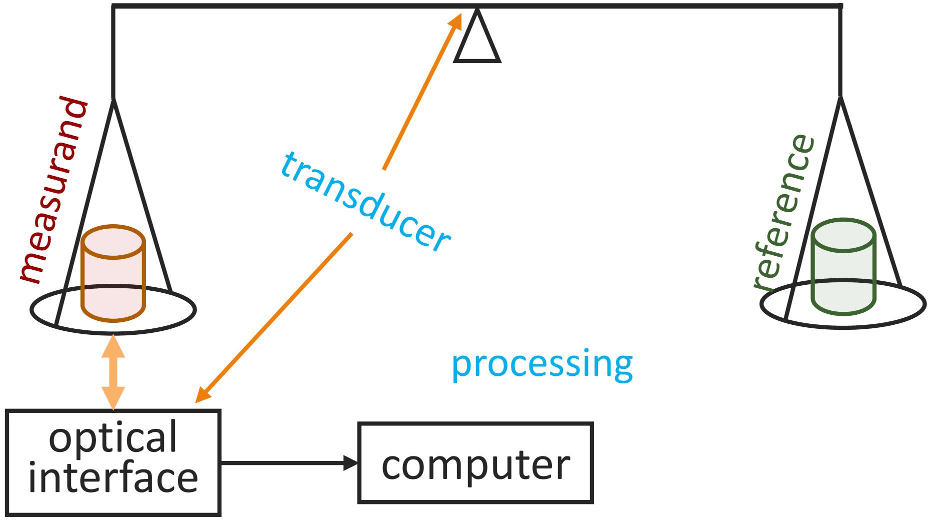
interface

ADC

processing

low-level precision
optical/electrical

beam balance



measurand

transducer

reference

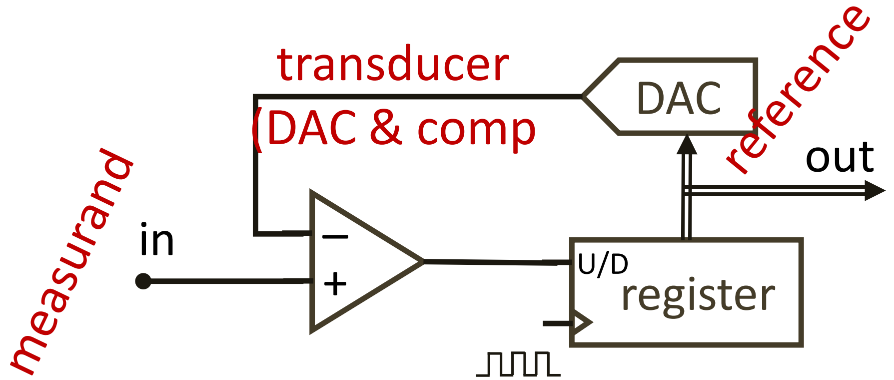
processing

optical interface

computer

caliper

Example, ADC



transducer
(DAC & comp

DAC

reference

out

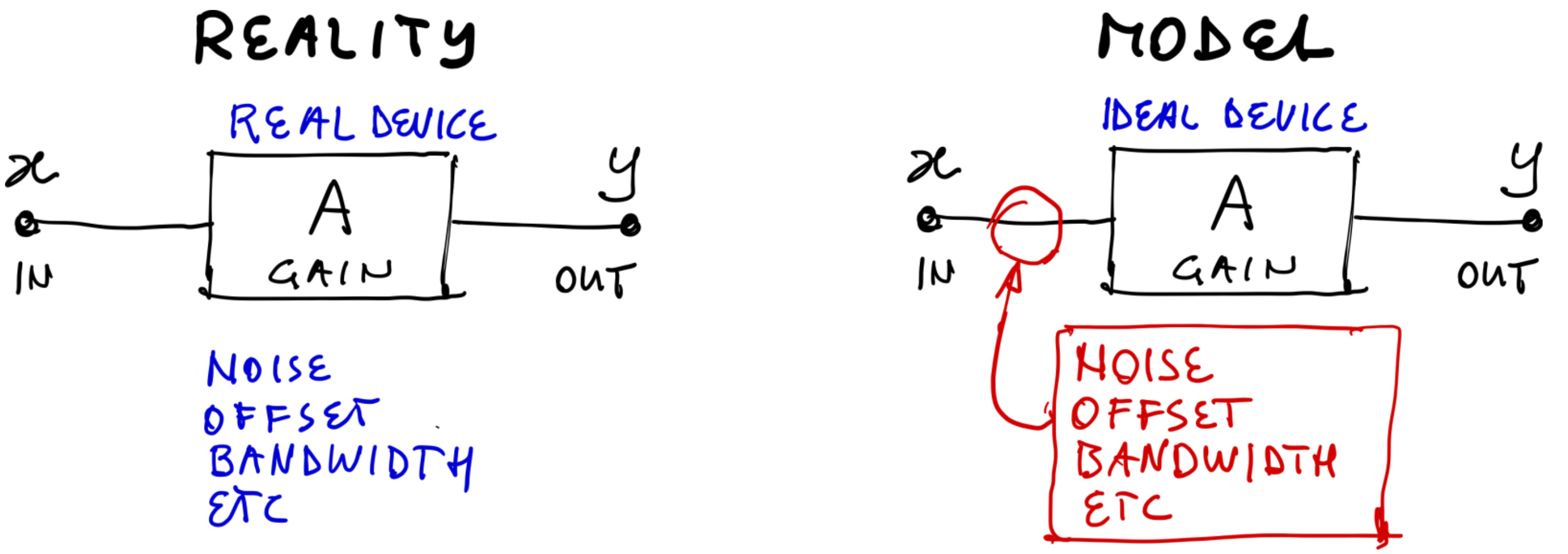
measurand

in

U/D
register

Up/Down counter
or Successive
Approximation
Register

Think with models



Logical sequence

- Identify the physical phenomena
- Order of magnitude first
- Block diagram
- Non-idealities
  - Referred at the input (preferred)
  - Referred at the output
- Information/energy flow
- Math at the end

Wheeler's First Moral Principle says, "Never make a calculation until you know the answer"

Sir John Archibald Wheeler, theoretical physicist (July 9, 1911 – April 13, 2008)

Thermal Noise

Planck constant $h = 6.02607015 \times 10^{-34}$ Js
Electron charge $e = 1.60207015 \times 10^{-19}$ C
Boltzmann constant $k = 1.380649 \times 10^{-23}$ J/K

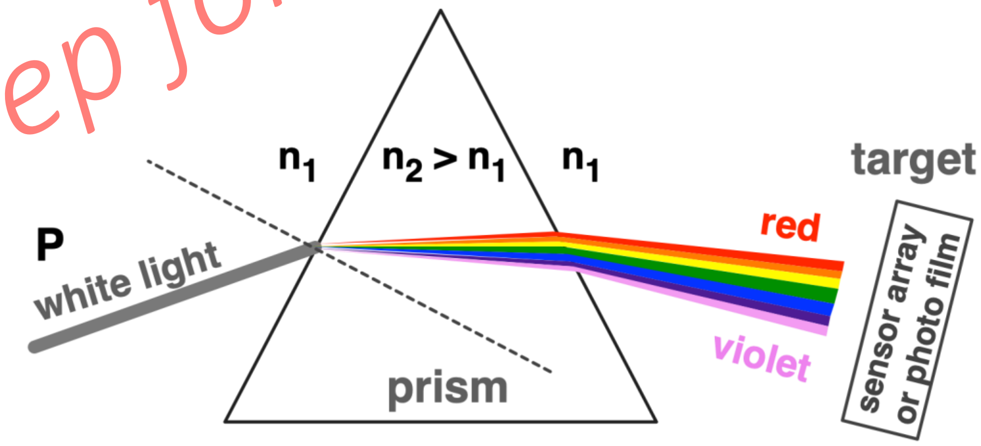
J. B. Johnson, Thermal Agitation of Electricity in Conductors, Phys Rev 32(1) p.97-109, July 1928

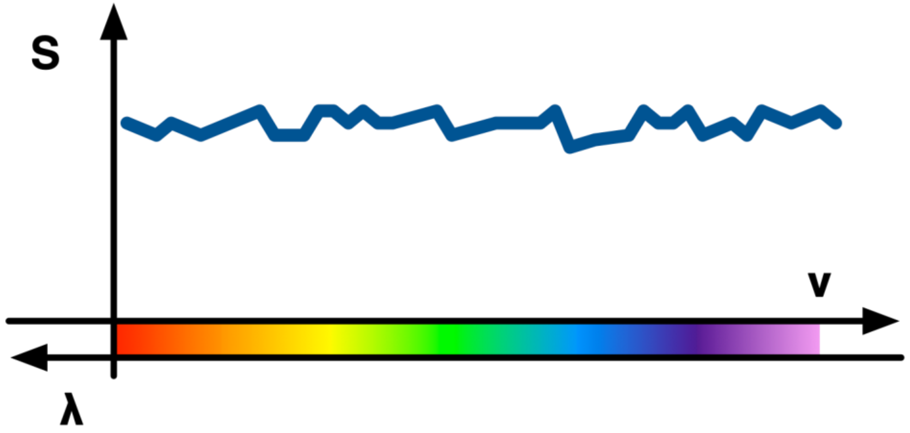
H. Nyquist, Thermal agitation of electric charges in conductors, Phys Rev 32(1) p.110-113, July 1928

The physical concept of spectrum

Keep for later

More precisely, Power Spectral Density





- The PSD is the distribution of power vs. frequency (power in 1-Hz bandwidth)
- The PS is the distribution of energy vs. frequency (energy in 1-Hz bandwidth)
- Power (energy) in physics is a square (integrated) quantity
- PSD –> W/Hz, or V^2/Hz, A^2/Hz, rad^2/Hz etc.

$$S_v(f) = \frac{\langle v_B^2(f)\rangle}{B}$$

Discrete: Δf is the resolution
Continuous: $\Delta f \to 0$

$$\frac{\text{average power in the bandwidth } B \text{ centered at } f}{\text{bandwidth } B}$$

The extended Planck law

Physical laws

Blackbody radiated energy

$$S(\nu) = \frac{h\nu}{e^{h\nu/kT} - 1} \quad \text{[W/Hz]}$$

At the receiver input

$$S(\nu) = h\nu + \frac{h\nu}{e^{h\nu/kT} - 1}$$

The additional $h\nu$ is the zero point energy

Nawrocki, Eq.1.13, Göbel-Siegner, Eq.2.10

Receiver

Thermal regime $h\nu \ll kT$

$$e^{h\nu/kT} \simeq 1 + h\nu/kT$$

$$S(\nu) = kT$$

Quantum regime $h\nu \gg kT$

$$e^{h\nu/kT} \gg 1$$

$$S(\nu) = h\nu$$

cutoff frequency

$$\nu_c = \frac{kT}{h}\ln(2)$$

Featured reading:

Chapter 1, W. Nawrocki, *Introduction to quantum metrology* 2nd ed, Springer 2019

Chapter 2, E. O. Göbel, U. Siegner, *The new International System of units*, Wiley VCH 2019

Cutoff frequency

$$\nu_c = \frac{kT}{h}\ln(2)$$

Reference	T, K	ν	λ
room	300	4.33 THz	69.2 µm
Liquid N_2	77	1.11 THz	270 µm
Liquid He	4.2	60.7 GHz	4.94 mm
^{3}He/^{4}He	0.01	144 MHz	2.08 m

POI – The dilution refrigerator

- ^{4}He is a boson
  - Superfluid at low temperature
- ^{3}He is a fermion
  - Pauli exclusion principle
  - Fermi liquid at low temperature
- Cooling process
  - Pre-cool the mixture to 1 K (cryocooler)
  - A capillary with large flow resistance cools to 0.5-0.7 K
  - The fluid is unstable
  - Phase separation is endothermal

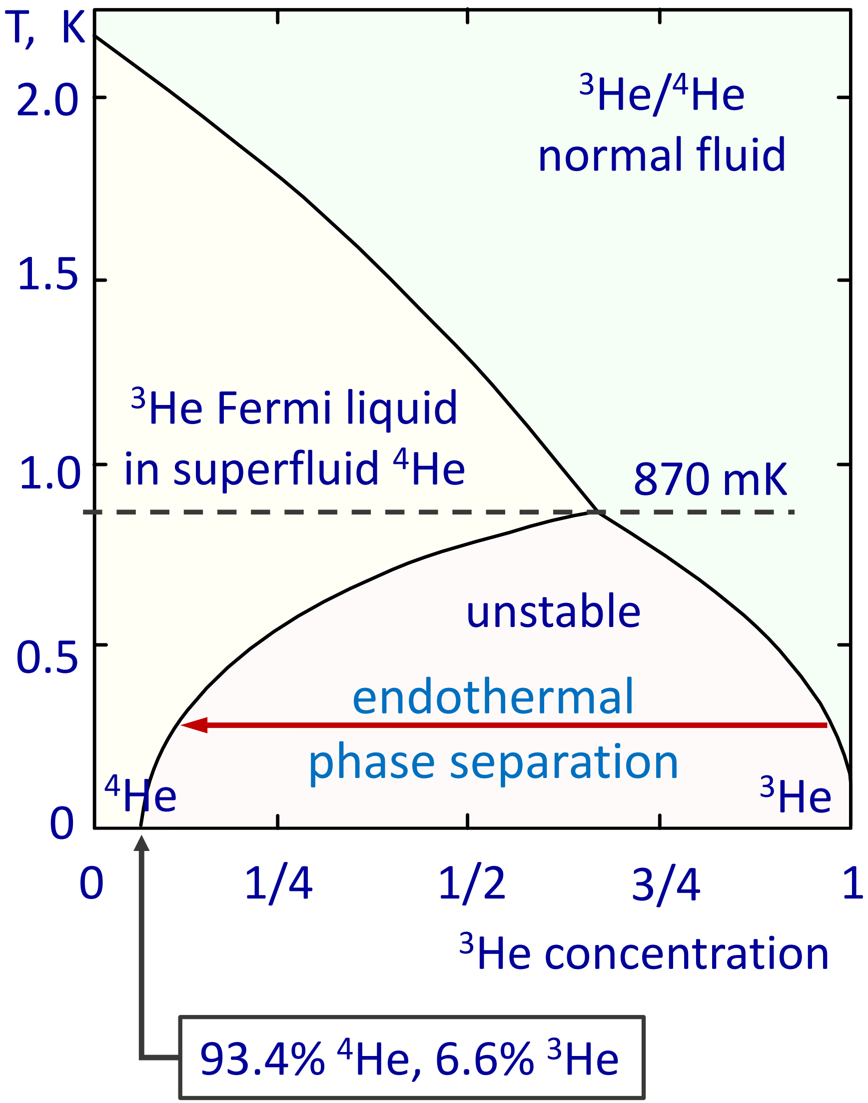


Photo E. Rubiola

Dilution refrigerator at the FEMTO-ST Institute

Theory: Heinz London, early 1950s
Implementation: 1964, Kamerlingh Onnes Lab, Leiden
H. K. Onnes (Nobel 1913) liquefied He (1908) and discovered the superconductivity of Hg (1911)

Featured reading:
Chapter 9, S. W. Van Sciver, Helium cryogenics 2nd ed., Springer 2012

The "Soul" of thermal noise

Thermal noise is blackbody radiation transmitted through an electrical line

It has two degrees of freedom, each has energy $kT/2$

electric and magnetic field

$$E_C = \frac{1}{2}CV^2 \rightarrow \frac{1}{2}kT$$

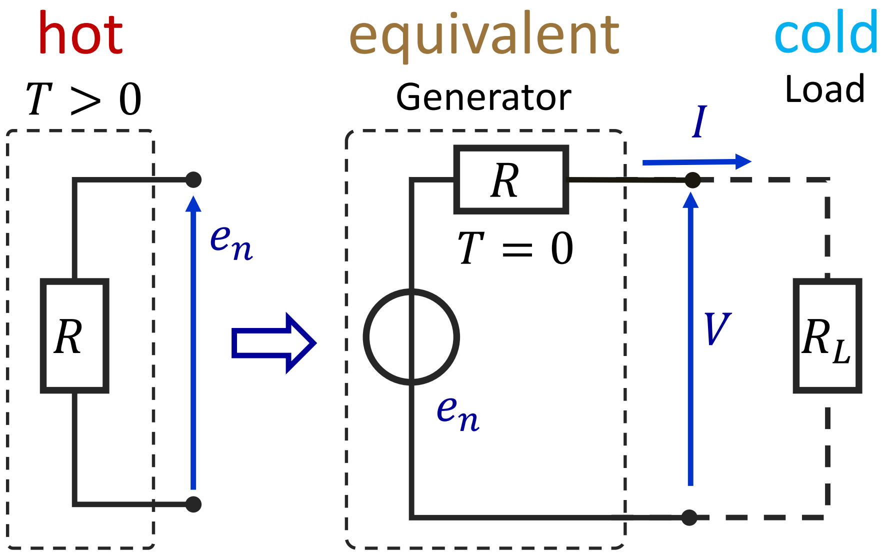
– or –

$$E_L = \frac{1}{2}LI^2 \rightarrow \frac{1}{2}kT$$

two polarization states

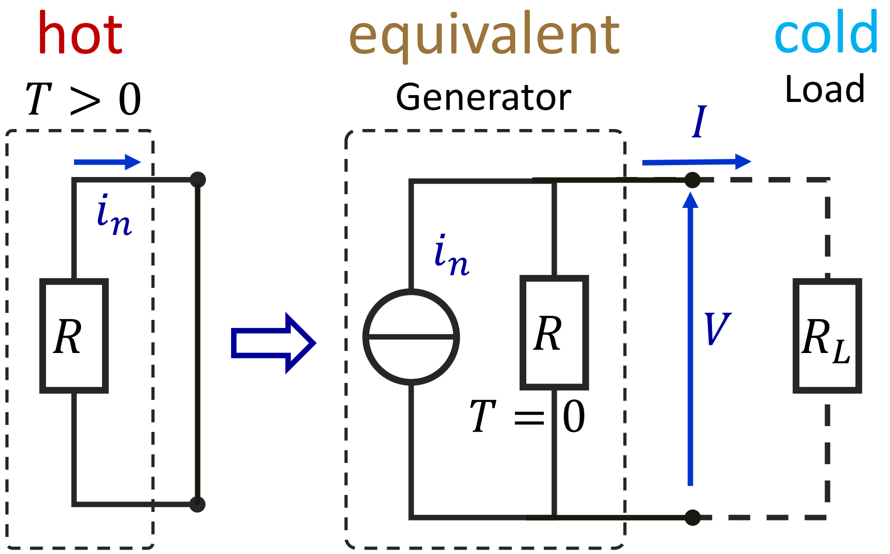
Thévenin and Norton models

Thévenin Model



Thermal EMF $V_G \to e_n \equiv \sqrt{S_v}$ $[\mathrm{V}/\sqrt{\mathrm{Hz}}]$

Norton Model



Thermal current $I_G \to i_n \equiv \sqrt{S_i}$ $[\mathrm{A}/\sqrt{\mathrm{Hz}}]$

Maximum power transfer $R_L = R_G$

$$V = \frac{1}{2} V_{\mathrm{open}} \qquad I = \frac{1}{2} I_{\mathrm{short}} \qquad P = \frac{1}{4} V_{\mathrm{open}} I_{\mathrm{short}}$$

Jargon: the *available* power/voltage/current is the $P/V/I$ delivered with $R_L = R$

Thermal noise

Terminated resistor (hot —> cold)

$S = kT$ W/Hz

$S_V = kTR$ V²/Hz

$S_I = kT/R$ A²/Hz

Two resistors at different temperature

$$S = k(T_2 - T_1)$$

$S_V = 4kTR$ Open circuit

$S_I = 4kT/R$ Short circuit

Bandwidth limited by cables / waveguide

Noise of a 50 Ω resistor

Reference	*T*, K	Available W/Hz	Open pV/√Hz	Short pA/√Hz
room	300	4.14×10^{-21}	910	18.2
T_0 (RF electronics)	290	4.00×10^{-21}	895	17.9
Dry ice (–78.5 °C)	194.7	2.69×10^{-21}	733	14.7
Liquid N_2	77	1.06×10^{-21}	461	9.22
Liquid He	4.2	5.80×10^{-23}	108	2.15
^{3}He/^{4}He	0.01	1.38×10^{-25}	5.25	0.105

J.B. Johnson JB,Thermal Agitation of Electricity in Conductors, Phys Rev 32(1) p.97-109, July 1928
H. Nyquist, Thermal agitation of electric charges in conductors, Phys Rev 32(1) p.110-113, July 1928

The Harry Nyquist’s article

Image user Quibik, Wikimedia

All schematics are from H. Nyquist 1928

Thermal equilibrium

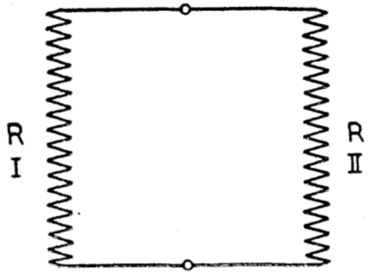


Fig. 1.

Thermal equilibrium also applies to any frequency (interval)

EMF E is a function of R, T and f only

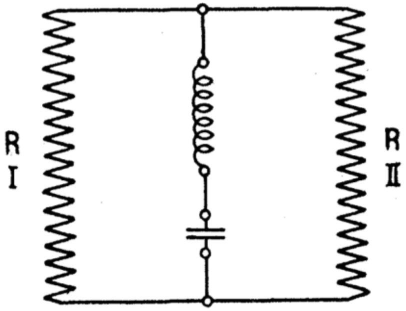


Fig. 2.

Loss-free, terminated electrical line

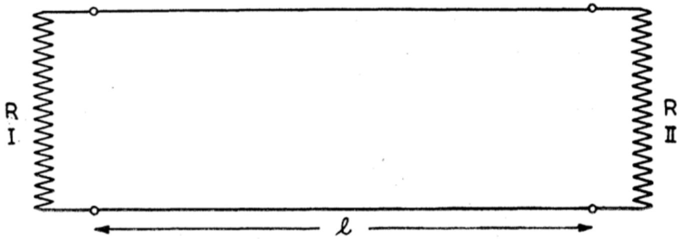


Fig. 3.

After thermal equilibrium, isolate the line (short at both ends).

Modes at $\nu = n\, c/\ell$

ν = frequency, c = velocity

Energy kT per mode

$$dE = 2\ell kT\, d\nu\, /\, c$$

Average power in frequency $d\nu$, and in time ℓ/c is $kT\, d\nu$

Extension to electrical circuits

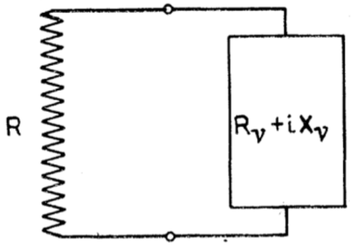


Fig. 4.

Energy per degree of freedom

$$h\nu/(e^{h\nu/kT} - 1)$$

instead of kT

Conclusion

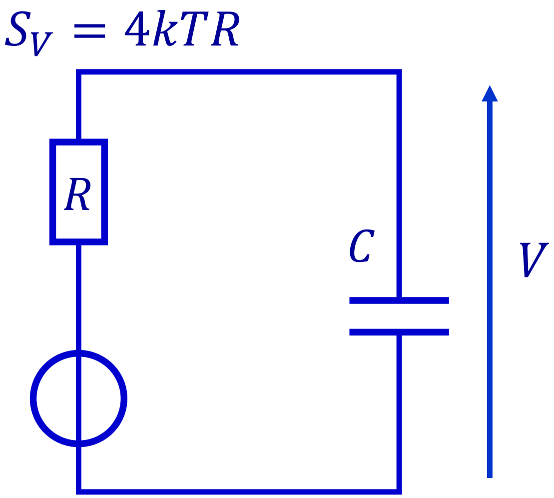
$$E_\nu^2\, d\nu = 4R_\nu\, h\, d\nu\, /\, (e^{h\nu/kT} - 1)$$

J. B. . Johnson, Thermal Agitation of Electricity in Conductors, Phys Rev 32(1) p.97-109, July 1928

H. Nyquist, Thermal agitation of electric charges in conductors, Phys Rev 32(1) p.110-113, July 1928

Thermal noise across a capacitor

Beware of CMOS gates and Track/Hold circuits



$$\langle V^2 \rangle = kT/C, \qquad \forall R$$

0.1 pF	200 µV	20 aC	125 e
1 pF	64 µV	64 aC	400 e
10 pF	20 µV	200 aC	1250 e
100 pF	6.4 µV	640 aC	4000 e
1 nF	2 µV	2 fC	12500 e

Proof (stat physics)

Capacitor $E = \frac{1}{2} C V^2$

The energy fluctuation per degree of freedom is

$$E = kT/2$$

at thermal equilibrium

Mean square fluctuation

$$C\Delta(V^2/2) = kT/2$$

Conclusion

$$\langle V^2 \rangle = kT/C$$

Sarpeshkar R, Delbruck T, Mead CA - White Noise in MOS Transistors and Resistors - Circuits and Devices, November 1993

Proof (circuit theory)

Voltage

$$S_V = 4kTR$$

Transfer function

$$|H(f)|^2 = \frac{1}{1 + (2\pi f RC)^2}$$

Mean square fluctuation

$$\langle V^2 \rangle = \int_0^\infty 4kTR\, |H(f)|^2 \, df$$

Conclusion, R cancels, and

$$\langle V^2 \rangle = kT/C$$

Trivial exercise

Shot Noise

Electron charge $e = 1.60207015 \times 10^{-19}$ C

W. Schottky, „Über spontane Stromschwankungen in verschiedenen Elektrizitatsleitern“, Annalen der Physik 362(23) p541-567, 1918 (in German). Get free pdf from Zenodo

Open access English translation "On spontaneous current fluctuations in various electrical conductors" by Martin Burkhardt, with additional editing by Anthony Yen

The exponential distribution

A cell emitting particles at random, at the average rate of ϕ events/s

In the literature we often find λ instead of ϕ, and x instead of t

Probability Density Function

PDF $$p(t;\phi) = \phi e^{-\phi t}, t \geq 0$$

Mean $\mu = 1/\phi$, Variance $\sigma^2 = 1/\phi^2$

$$\mu = \int t\, p(t;\phi)\, dt = 1/\phi$$

$$\sigma^2 = \int (t-\mu)^2\, p(t;\phi)\, dt = 1/\phi^2$$

Properties

Memoryless $\mathbb{P}\{T > s + t | T > s\} = \mathbb{P}\{T > t\}$

T is the waiting time

- Statistically, T is the same starting at 0 or at s, if the particle did not show up
- Maximum differential entropy —> maximum entropy for a given μ

This describes “emissions” in physics

- Electrons and holes in a junction
- Photons
- Radioactive decay (assuming that the nuclei are not lost)

Featured reading:

W. Feller, *Introduction to probability theory and its applications*, 2nd ed, Wiley. Vol.I, 1957, vol.II, 1970

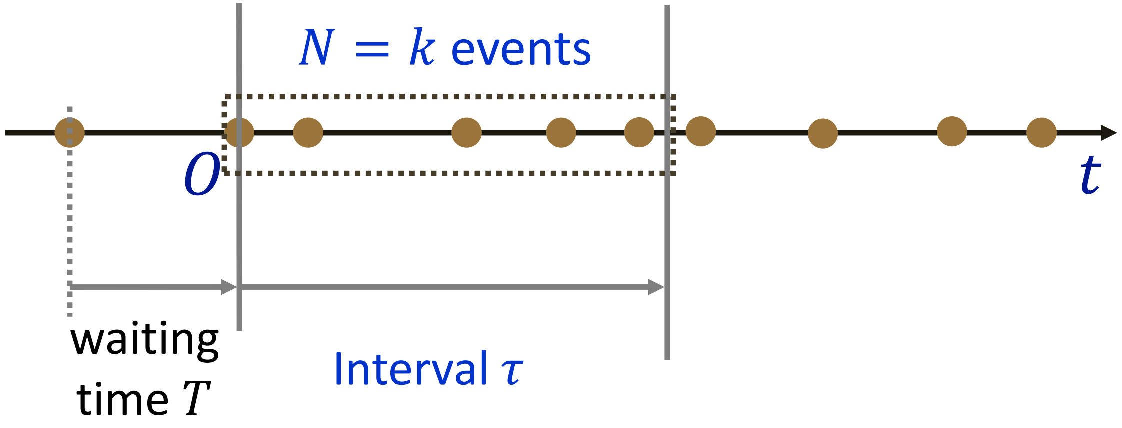
Vol. 1, Sec. XVII-6 provides a proof that in a memory-less process, the tail of the distribution has to be of the form $u = \exp -\lambda t$ (or zero), and nothing else. See also vol.II, Sec. I-3

Homogeneous Poisson process

An ensemble of memoryless and statistically independent cells emitting at random at the average rate (flux) of ϕ events/s

$$\mathbb{P}\{N(\tau)=k\}=\frac{(\phi\tau)^k}{k!}e^{-\phi\tau}$$

$\mathbb{P}$ is the probability that the number N of particles emitted from time 0 to τ equals k



My notebook vol. XXIII p. 49

Properties

average

$\mathbb{E}\{N(\tau)\}=\phi t$ written as $\boxed{\mu=\phi\tau}$

variance

$\mathbb{E}\{[N(\tau)-\mu]^2\}=\phi t$ $\boxed{\sigma^2=\phi\tau}$

signal-to-noise ratio

$SNR=\sigma/\mu$ $\boxed{SNR=\sqrt{N}}$

physical meaning of ϕ

$\lim_{t\to\infty}\frac{N(t)}{t}=\phi$ average no of events / time, flux in the case of particle emission

W. Feller, Introduction to Probability Theory and Its Applications, vol.II, 2nd ed., Wiley 1970

Shot noise

Charge

e	$\mathbb{E}(Q) = \phi\tau e$	[C]
e^2	$\mathbb{V}(Q) = \phi\tau e^2$	[C^2]
$e^2\tau$	$\mathrm{S}_Q(f) = 2\phi\tau^2 e^2$	[C^2/Hz]

Current

e/τ	$\mathbb{E}(I) = \phi e$	[A]
e^2/τ^2	$\mathbb{V}(I) = \phi\tau(e/\tau)^2$	[C^2]
e^2/τ	$\mathrm{S}_I(f) = 2\phi\tau^2(e^2/\tau^2)$	
	$= 2\phi e^2 = 2eI$	[A^2/Hz]

Photon energy

$h\nu$	$\mathbb{E}(Q) = \phi\tau h\nu$	[J]
$(h\nu)^2$	$\mathbb{V}(Q) = \phi\tau(h\nu)^2$	[J^2]
$(h\nu)^2\tau$	$\mathrm{S}_Q(f) = 2\phi\tau^2(h\nu)^2$	[J^2/Hz]

Photon power

$h\nu/\tau$	$\mathbb{E}(I) = \phi h\nu$	[W]
$(h\nu)^2/\tau^2$	$\mathbb{V}(I) = \phi\tau(h\nu/\tau)^2$	[W^2]
$(h\nu)^2/\tau$	$\mathrm{S}_I(f) = 2\phi\tau^2[(h\nu)^2/\tau^2)$	
	$= 2\phi(h\nu)^2$	[W^2/Hz]

Quantum Limit

Planck constant $h = 6.02607015 \times 10^{-34}$ Js
Electron charge $e = 1.60207015 \times 10^{-19}$ C
Boltzmann constant $k = 1.380649 \times 10^{-23}$ J/K

This section is based upon

E. O. Göbel, U. Siegner, The New International System of Units (SI), Wiley VCH 2019

See also

M. Gläser, M. Kochsiek (Ed.), Handbook of Metrology vol.1-2, Wiley VCH 2010

V. B. Braginsky, F. Ya. Khalili, Quantum Measurement, Cambridge 1992

Fundamental quantum limit

Photon energy

$$E = h\nu$$

Heisenberg Principle:
The minimum action H is

$$H \gtrsim h$$

If p and x are momentum and position,

$$\Delta x \Delta p \geq \frac{1}{2}\hbar$$

Planck constant
$h = 6.02607015 \times 10^{-34}$ Js
(exact)

Application to the measurement

Energy extracted from the system in the time τ

$$E \gtrsim h/\tau \quad \text{or} \quad E \gtrsim hB$$

Measurement bandwidth

⇦ $B = 1/\tau$

Featured reading: Chapter 2, E. O. Göbel, U. Siegner, The New International System of Units, Wiley VCH 2019
Also: V. B. Braginsky, F. Ya. Khalili, Quantum Measurement, Cambridge 1992

Quantum limit in the capacitor

$$E \gtrsim h/\tau$$

Voltage

Energy

$$\boxed{\frac{1}{2}CV^2} \gtrsim \frac{h}{\tau}$$

$$V \gtrsim \sqrt{\frac{2h}{\tau C}}$$

Use large C and τ

$C = 1.5$ nF, $\tau = 10$ ms

$V = 9.4$ pV

Charge

Energy

$$\boxed{\frac{1}{2}\frac{Q^2}{C}} \gtrsim \frac{h}{\tau}$$

$$Q \gtrsim \sqrt{\frac{2hC}{\tau}}$$

Use small C and large τ

$C = 2$ pF, $\tau = 10$ ms

$Q = 5.15 \times 10^{-22}$ C

$Q \ll e = 1.6 \times 10^{-19}$ C

Featured reading: Chapter 2, E. O. Göbel, U. Siegner, The New International System of Units, Wiley VCH 2019

Quantum limit in the inductor

$$E\tau \gtrsim h$$

Current

Energy

$$\boxed{\frac{1}{2}LI^2} \gtrsim \frac{h}{\tau}$$

$$I \gtrsim \sqrt{\frac{2h}{\tau L}}$$

Use large τ and L

$L = 200$ mH, $\tau = 100$ ms

$I = 25.7$ aA

Flux

Energy

$$\boxed{\frac{1}{2}\frac{\Phi^2}{L}} \gtrsim \frac{h}{\tau}$$

$$\Phi \gtrsim \sqrt{\frac{2hL}{\tau}}$$

Use small L and large τ

$L = 2.5$ nH, $\tau = 100$ ms

$\Phi = 5.8 \times 10^{-21}$ Wb

$\mu_0 \simeq 1.257\ \mu\text{H/m}$

$L = \mu_0 \ell \rightarrow \ell = 2\ \underline{\text{m}}\text{m}$

$\Phi_0 = \frac{h}{2e} = 2.0678 \times 10^{-15}$ Wb

Featured reading: Chapter 2, E. O. Göbel, U. Siegner, The New International System of Units, Wiley VCH 2019

Quantum limit in the resistor

$$E\tau \gtrsim h$$

Voltage

Energy

$$\boxed{\frac{V^2}{R}\tau} \gtrsim \frac{h}{\tau}$$

$$V \gtrsim \sqrt{hR}\,\frac{1}{\tau}$$

Current

Energy

$$\boxed{RI^2\tau} \gtrsim \frac{h}{\tau}$$

$$I \gtrsim \sqrt{\frac{h}{R}\frac{1}{\tau}}$$

Use small R and large τ

$R = 50\ \Omega,\ \tau = 100$ ms

$V = 1.82$ fV

Use large R and τ

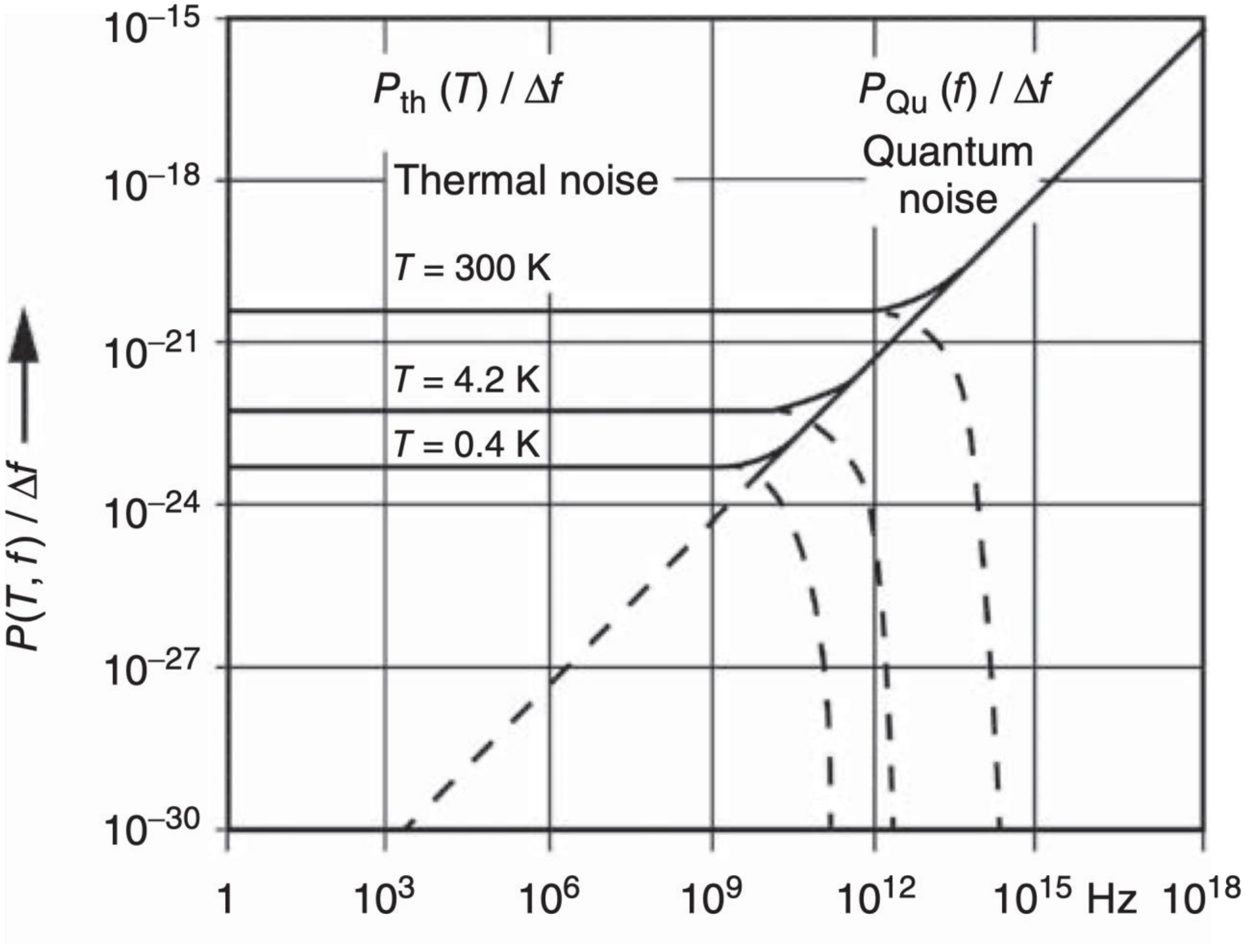
$R = 1\ \mathrm{M}\Omega,\ \tau = 100$ ms

$I = 2.57 \times 10^{-19}$ A

$e = 1.6 \times 10^{-19}$ C

Featured reading: Chapter 2, E. O. Göbel, U. Siegner, The New International System of Units, Wiley VCH 2019

Thermal vs quantum noise



This figure is from

Siebert, B.R.L. and Sommer, K.D. (2010) in *Uncertainty in Handbook of Metrology*, vol. **2** (eds M. Gläser and M. Kochsiek), Wiley-VCH Verlag GmbH, Weinheim, pp. 415–462.

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Lecture 1 ends here

Lecture 2

Scientific Instruments & Oscillators

Spring 2022

Lectures for PhD Students and Young Scientists

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Contents

- Flicker noise
- General instrument architecture
- Noise in electronic devices

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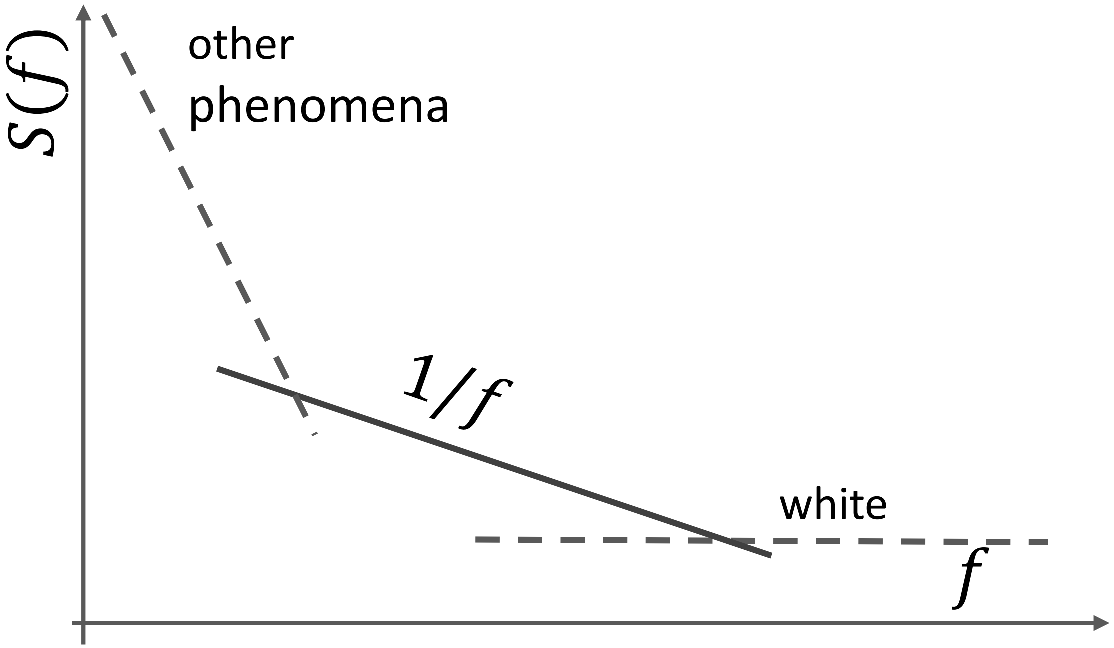
home page http://rubiola.org

Flicker $(1/f)$ Noise

Ubiquitous phenomenon in science and technology

Flicker ($1/f$) noise



- Extremely weak noise phenomenon
- A major issue in spectral analysis
- Relevant in cryogenic nanodevices and qbits
- Resolution cannot be improved by increasing the measurement time

- Observed in a large variety of phenomena (conductance, semiconductors, vacuum tubes, music and radio broadcasting, Internet, pulsars, squids, earthquakes, fractals)
- Electronics, exact $1/f$ slope up to 8 decades
- Other fields, $1/f^{\alpha}$, $\alpha = 0.5 \ldots 1.5$
- Discovered by Johnson, 1925
- Studied in carbon microphones and in the fluctuation of resistivity, >1930
- Well explained in some cases (magnetics…)
- No unified theory

Q: Is this Type A or Type B evaluation of uncertainty?

Integrated flicker noise is extremely small

How small the $1/f$ noise can be?

$$\sigma^2 = \int_a^b \frac{1}{f}\, df = \ln\left(\frac{b}{a}\right)$$

Let’s consider the widest, craziest frequency range

$$a = \frac{1}{A_U}$$

Age of Universe

$$b = \frac{1}{2\pi\tau_P}$$

Planck time (Gauss)

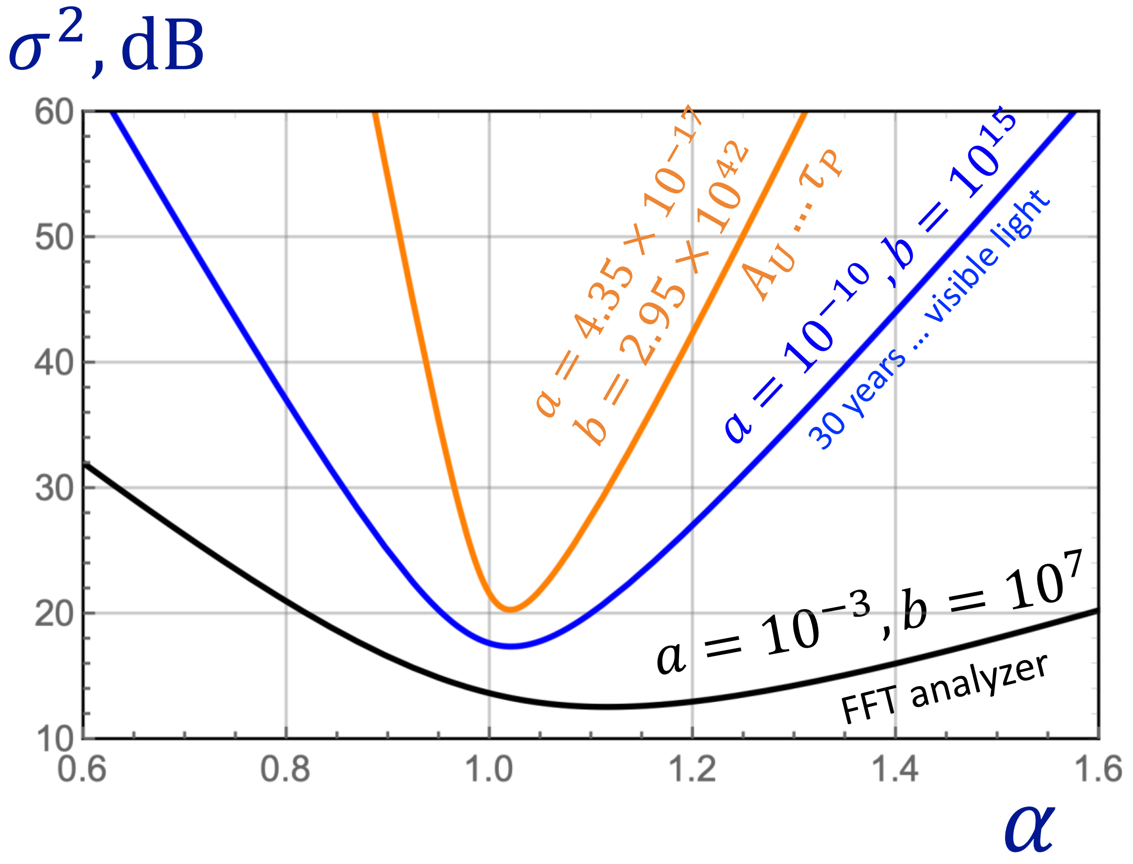
$$A_U = 4.35 \times 10^{17}\text{s} \ \ (13.8 \text{ By})$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \simeq 5.39 \times 10^{-44} \text{ s}$$

$$\ln\left(\frac{b}{a}\right) = \ln\left(\frac{1/2\pi t_P}{1/A_U}\right) = 138.4 \qquad (21.4\text{dB})$$

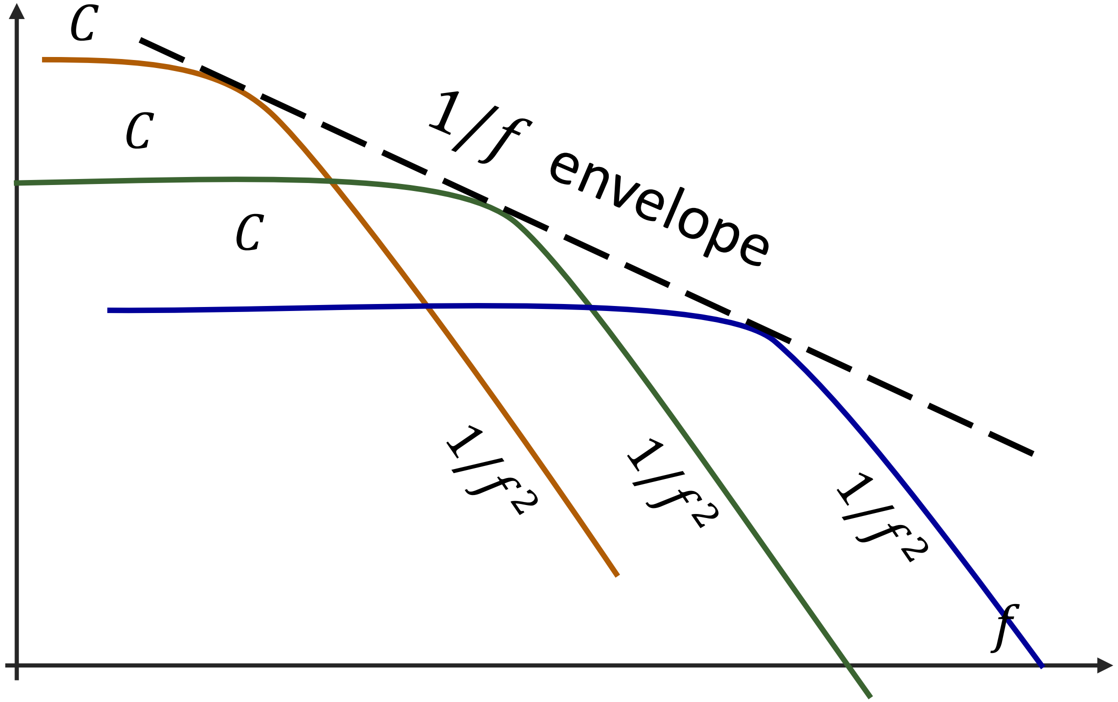
Integrated $1/f^{\alpha}$ noise is small even for $\alpha \neq 1$

$$\sigma^2 = \int_a^b \frac{1}{f^{\alpha}} df$$



Distribution of relaxation times

Uniform (random) distribution of time constants on a log-log scale



Featured reading: E. Milotti, $1/f$ noise, a pedagogical review, arXiv.physics 0204033, April 2002

1/f noise and FD theorem

Flicker $(1/f)$ dimensional fluctuation is powered by thermal energy

Debye-Einstein theory for heat capacity

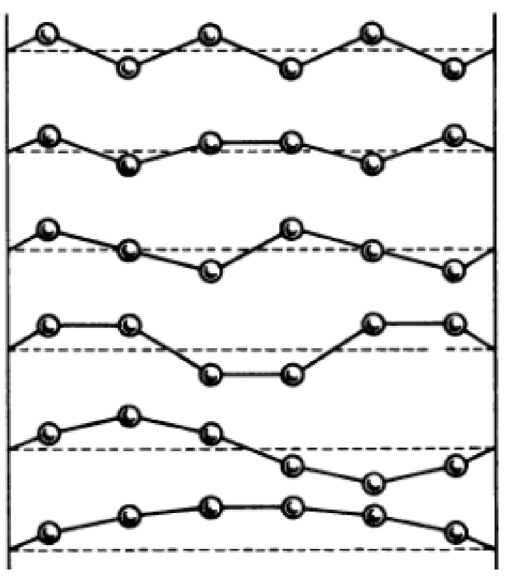


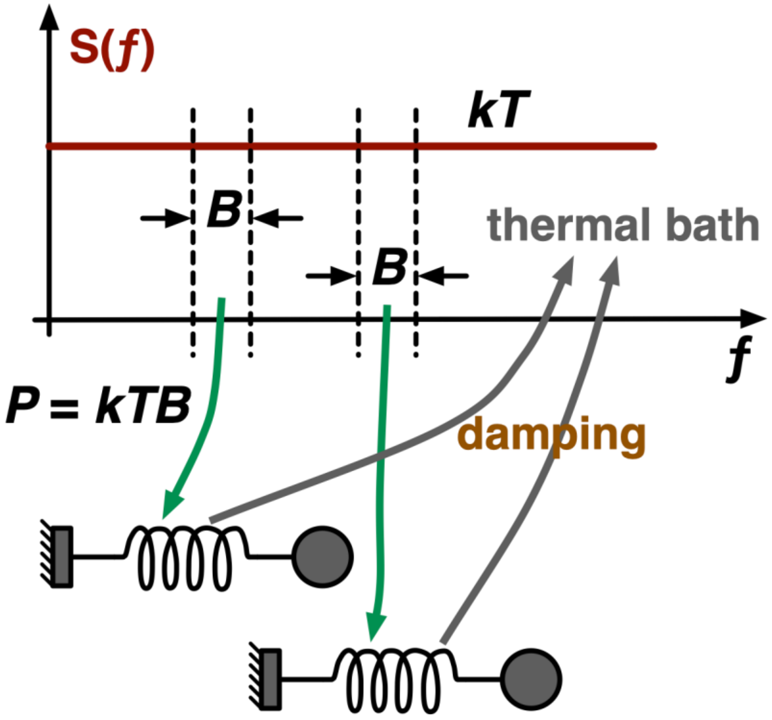
Figure 8-(b) from A. Holden A, The nature of solids, ©1965 Columbia University Press. Dover unabridged, 1992

A single theory explains

- Heat capacity
- Thermal expansion
- Elasticity

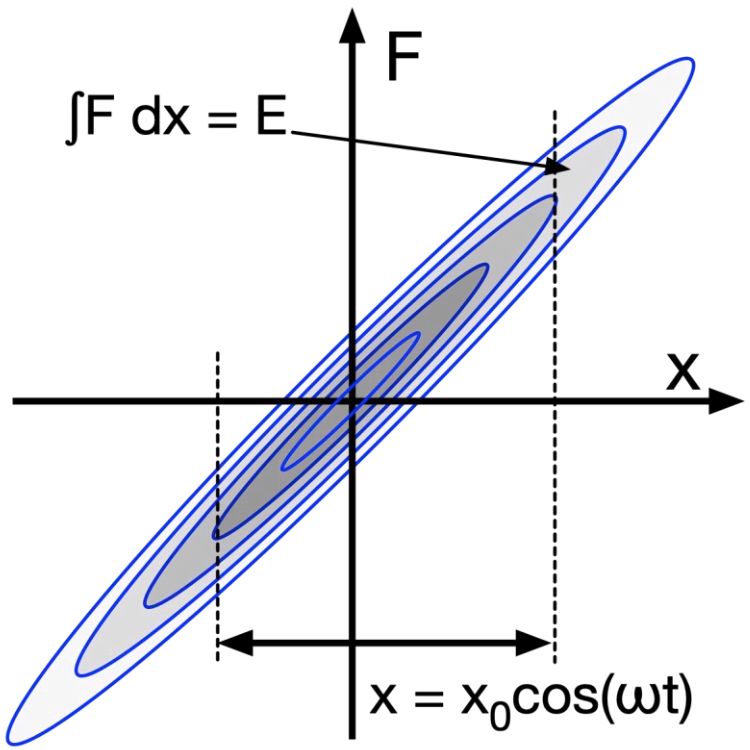
… and their fluctuations

Fluctuation Dissipation theorem in a nutshell



Thermal equilibrium applies to all portions of spectrum

Thermal $1/f$ from structural dissipation



There is no viscous dissipation in solids

Dissipation is structural (hysteresis)

Structural dissipation

micro/nanoscale, instantaneous

Dissipated energy $E = \int F dx$

Small vibrations

The hysteresis cycle keeps the aspect ratio

$E \propto x_0^2$ Energy lost in a cycle

Thermal equilibrium

$P = kT$ in 1 Hz BW

$P \propto kT x_0^2$

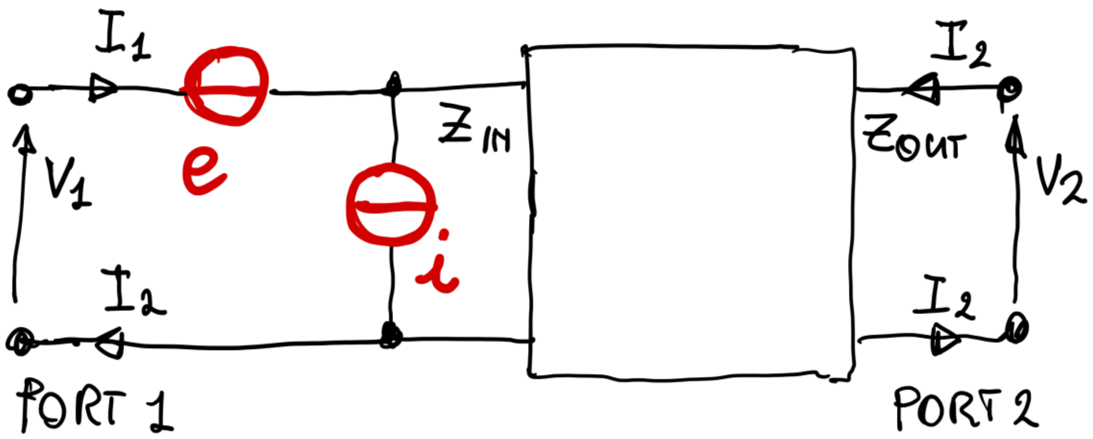
$x_0^2 \propto 1/f$ —> flicker

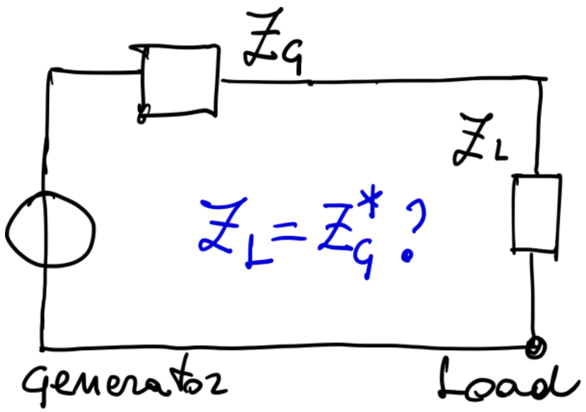
Bibliography about flicker

- C. J. Christiansen, G. L. Pearson, Spontaneous Resistance Fluctuations in Carbon Microphones and Other Granular Resistances, BSTJ 15(2) p.197-223, April 1937. Arguably, the discovery of flicker.
- F. N. Hooge, $1/f$ noise is no surface effect, Phis Lett 29(3) p.139-140, 21 April 1969. Classical article.
- D. J. Levitin, P. Chordia, V. Menon, Musical Rythm Spectra from Bach to Joplin Obey to $1/f$ Power Law, Proc. Nat. Academy of Science 109(10) p.716-3720, February 2012.
- A. L. Mcwhorter, $1/f$ Noise and Germanium Surface Properties, Proc. Semiconductor Surface Physics p.2017-228, June 1956. Classical article.
- E. Milotti, $1/f$ noise, a pedagogical review, arXiv.physics 0204033, April 2002.
- Paladino E et al., $1/f$ Noise, Implications for solid-state quantum information, Rev Modern Phys 86, April-June 2014
- Numata K, Kemery A, Camp J, Thermal-noise limit in the frequency stabilization of lasers with rigid cavities, Phys Rev Lett 93(25) 250602, December 2004.
- P. R. Saulson, Thermal Noise in Mechanical Experiments, Phys Rev D 42(8), October 1990.
- A. van der Ziel, Unified Presentation of $1/f$ Noise in Electronic Devices: Fundamental $1/f$ sources, Proc IEEE 76(3), March 1988
- L. K. J. Vandamme, G. A. Trefan, A review of $1/f$ noise in bipolar transistors, Fluct Noise Lett 1(4) 2001
- M. B. Weissman, $1/f$ noise and other slow, nonexponential kinetics in condensed matter, Rev Modern Phys 60(2) p.537-571, April 1988

The Rothe Dahlke model

e and i are the rms noise in 1 Hz bandwidth





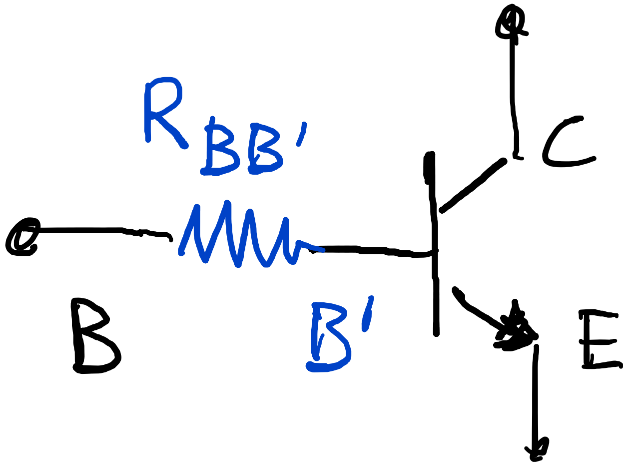
Noise is modeled as a voltage generator $e(t)$ and a current generator $i(t)$

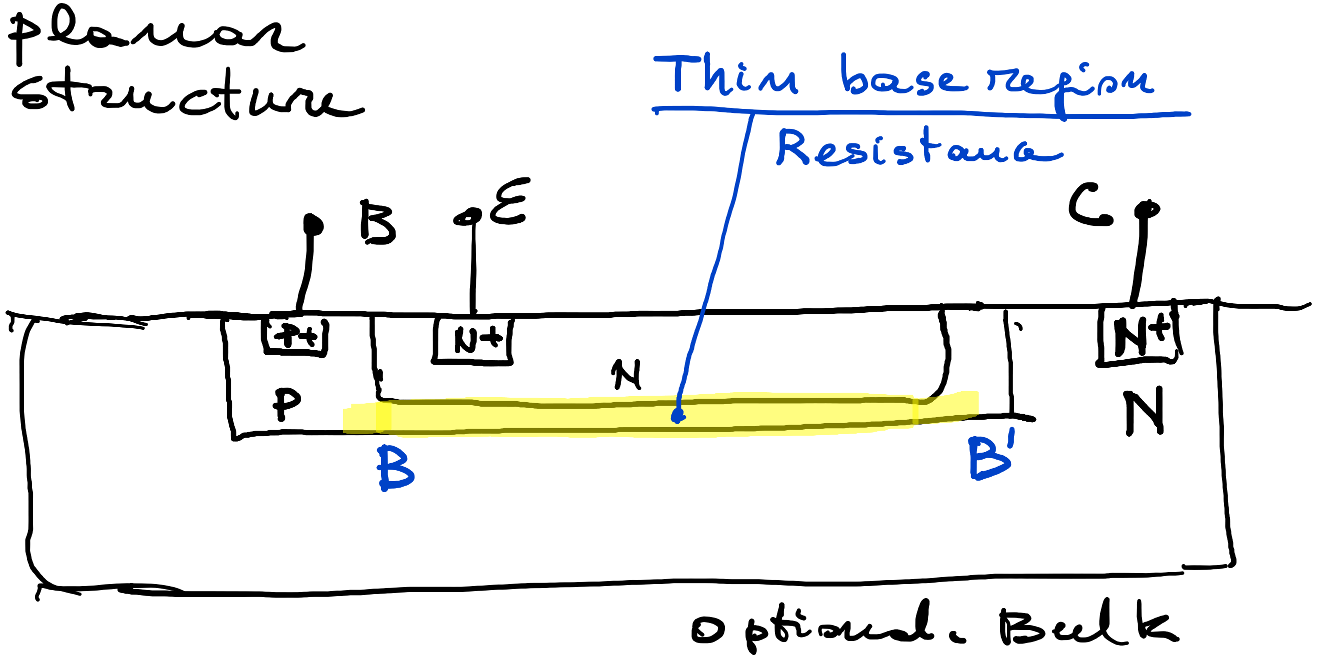
Consequences

- The golden rule $Z_L = Z_G^*$ is broken
- Three different impedance-matching criteria at Port 1 (the device is the load)
  - Lowest noise: $Z_G = e_n/i_n$
  - Maximum power: $Z_L = Z_G^*$
  - Highest Signal-To-Noise Ratio (SNR): something in between

H. Rothe, W. Dahlke, Theory of Noisy Fourpoles, Proc IRE 44(6) p.811-818, June 1956
H. A. Haus, R. B. Adler, Circuit theory of linear noisy networks, John Wiley & Sons 1959

Noise in bipolar transistors





White noise

e_n –> thermal noise in $R_{BB'}$

$(500\ \Omega \rightarrow 2.9\ \mathrm{nV}/\sqrt{\mathrm{Hz}})$

i_n – shot noise of I_B (note that $I_B \ll I_C$)

$(1\ \mu\mathrm{A} \rightarrow 0.57\ \mathrm{pA}/\sqrt{\mathrm{Hz}})$

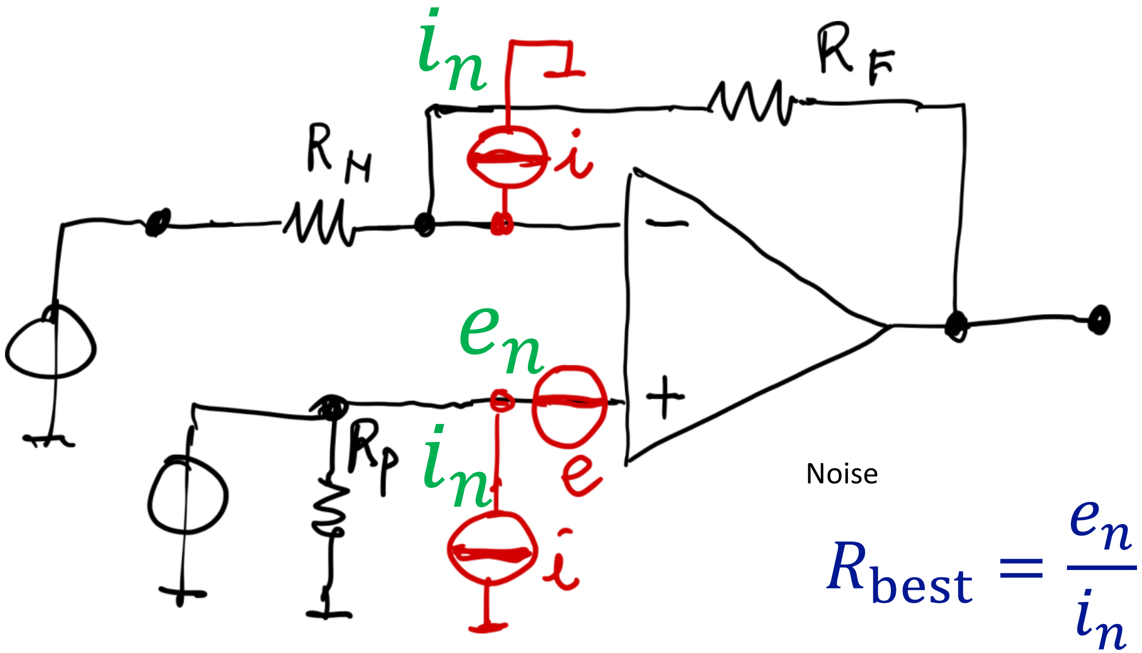
Flicker noise

Mainly the $1/f$ of the base current

H. K. Gummel and H. C. Poon, "An integral charge control model of bipolar transistors", Bell Syst. Tech. J. 49, pp. 827-852, 1970

Noise in operational amplifiers

$a \oplus b = (1/a + 1/b)^{-1}$



Noise

$$R_{\text{best}} = \frac{e_n}{i_n}$$

(+thermal)

Need to design precision electronics?

- D. Feucht, Analog Circuit Design Series, 4 volumes, SciTech 2010, ISBN 978-1-891121-XY-Z (old school but great)
- S. Franco S, Design with operational amplifiers and analog integrated circuits 4ed, McGraw Hill 2015, ISBN 978-0-07-802816-8 (best for designing with operational amplifiers)
- P. Horowitz, W. Hill, The Art of Electronics 3ed, Cambridge 2015, ISBN 978-0-521-80926-9 (Bible of instrument design, physical insight)
- Tietze U, Schenk C, Gamm E - Electronic Circuits 2ed - Springer 2007, ISBN 978-3-540-78655-9

Noise resistance

$$R_{eq} = R_P + (R_N \oplus R_F)$$

Voltage

$$V = V_{OS} + R_P I_P - (R_N \oplus R_F) I_N$$

Split I_N and I_P into offset and bias, $I_{OS} \pm \frac{1}{2} I_B$

Bias $I_B = \frac{1}{2}(I_P - I_N)$, Offset $\mathrm{I_{OS}} = I_P - I_N$

Total effect

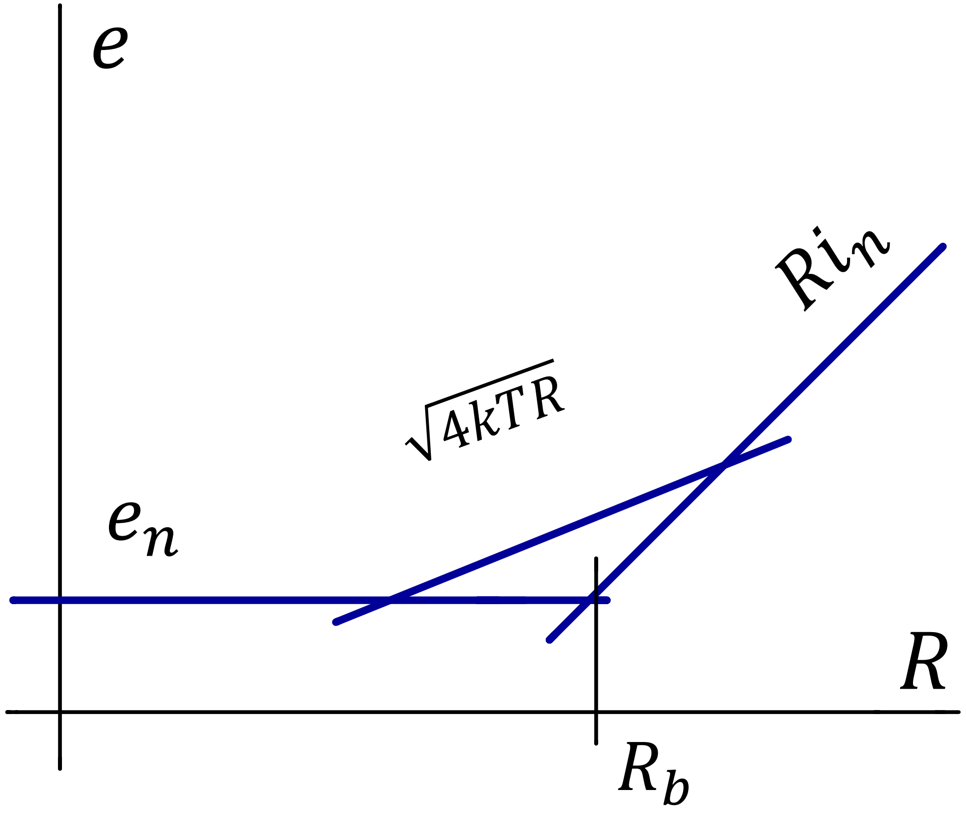
$$V = V_{OS} +$$
$$+ \frac{1}{2}[R_P - (R_F \oplus R_N)] I_B +$$
$$+ [R_P + (R_N \oplus R_F)] I_{OS}$$

Obvious extension to noise

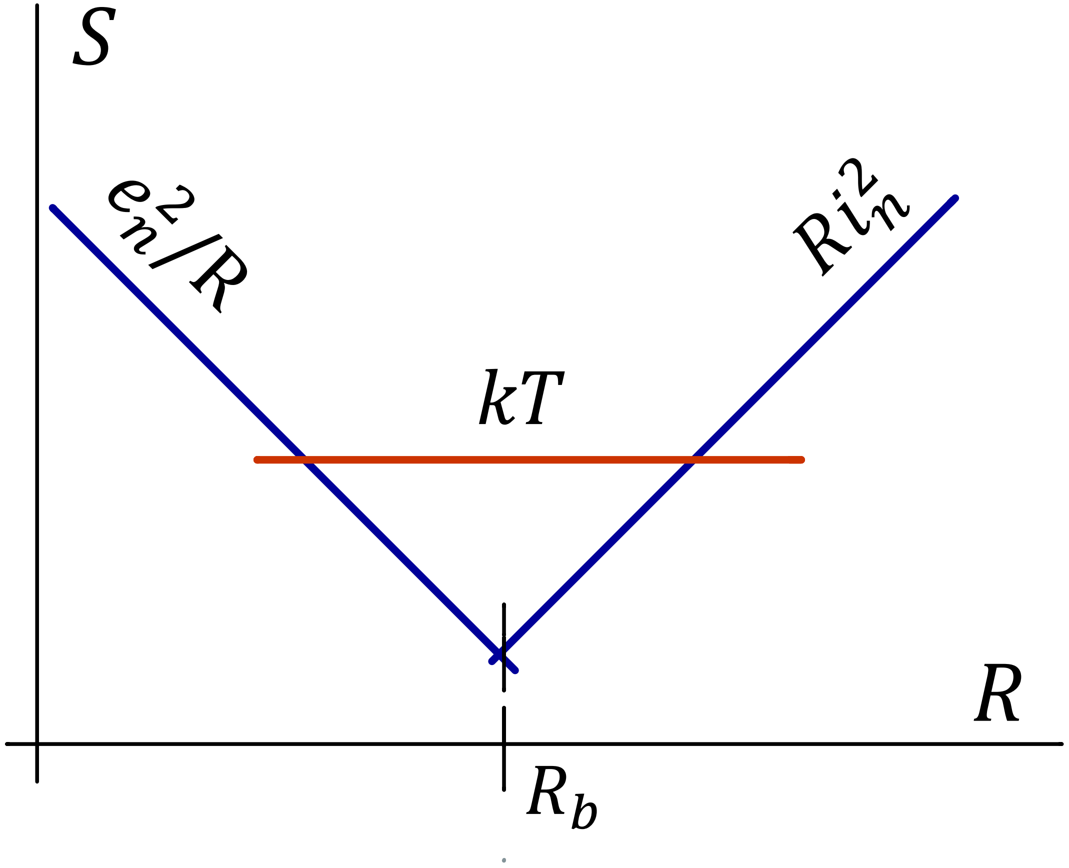
$$V^2 = \sum_i V_i^2$$

Noise power vs R

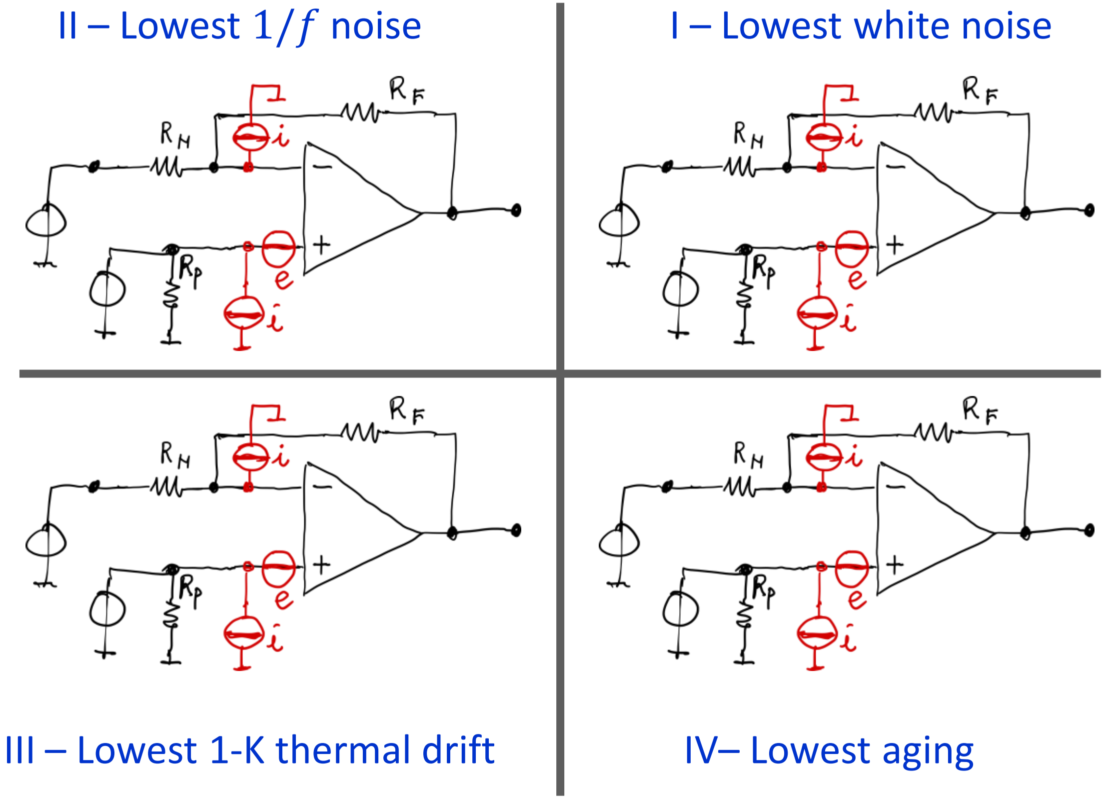
How it is shown



What it means



The Enrico's low-level near-DC design



II – Lowest $1/f$ noise

I – Lowest white noise

III – Lowest 1-K thermal drift

IV– Lowest aging

- Try a few designs based on different criteria
- Give a score to each feature
- Don't look down at not-so-important parameters
- Let beginners believe that only a small number of parts are important in precision electronics

Featured reading, low white noise and low $1/f$ noise design

E. Rubiola, F. Lardet-Vieudrin, Low flicker-noise amplifier for 50 Ω sources, Rev. Scientific Instruments 75(5) p.1323-1326, May 2004

Featured reading, random walk and aging

E. Rubiola, C. Francese, A. De Marchi, Long-Term Behavior of Operational Amplifiers, IEEE T IM 50(1) p.89-94, February 2001

Special cases

Extremely low current

- Charge amplifier (AD549, bias ≈100 e/s rms)
- Don't assume that insulators do insulate
- Prevent leakage with layout rules and guarding
- Narrow bandwidth only
- Polymers take in vibes (piezoelectricity)

Extremely low voltages

- Chopper (switching) amplifier (AD8628 ≈2 nV/K thermal)
- Bandwidth limited by the chopper frequency
- Thermocouples (Seebeck effect) are everywhere (soldering alloy, O_2 in Cu cables)
- Polymers take in vibes (electrostriction/piezoelectricity)

Highest gain accuracy

- Use Vishay resistor pairs (thermally compensated ratio)
- Unsuspected effects
    - Common mode rejection extremely critical
    - Open loop gain of OAs affects the accuracy
    - Thermal feedback inside OAs due to the power dissipated in the output stage
    - …and others

Lowest noise

- The choice of all resistances depends on e_n and i_n
- Bipolar transistor are better than field-effect transistors
- The design for lowest white or lowest 1/ƒ is not the same
- PNP amplifiers feature lower 1/ƒ noise

Photodiode signal

- The photodiode has high output impedance (current generator with a capacitance in parallel)
- Special design rules (Read J. G. Graeme, Photodiode amplifiers, McGraw Hill 1995, ISBN 0-07-024247-X)

Highest speed (video amplifier)

- Current feedback amplifiers (CFA, the bandwidth does not decrease with the gain)
- Higher noise

Highest speed (video amplifier) without CFAs

- Takes OPAs with extremely high gain-bandwidth product
- Self oscillations difficult to prevent (simulation must include L and C associated to the PCB

Featured reading: P. Horowitz, W. Hill, The Art of Electronics, 3rd ed, Cambridge 2015

Low-frequency shielding

Electric shielding is poor

- Skin effect

$$\delta = \sqrt{\frac{2\rho}{\omega\mu}} \quad \text{for } \omega \ll 1/\rho\epsilon$$

In Copper

9.2 mm at 50 Hz

2.06 mm at 1 kHz

ω = angular frequency

ρ = resistivity

μ = magnetic permeability

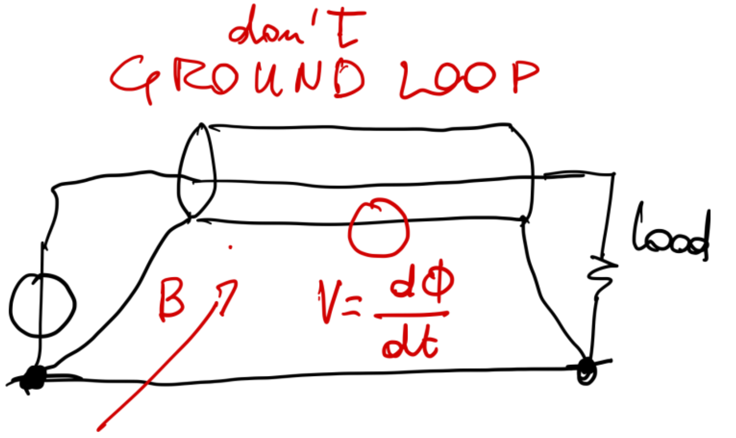
ϵ = electric permittivity

Magnetic shield is effective

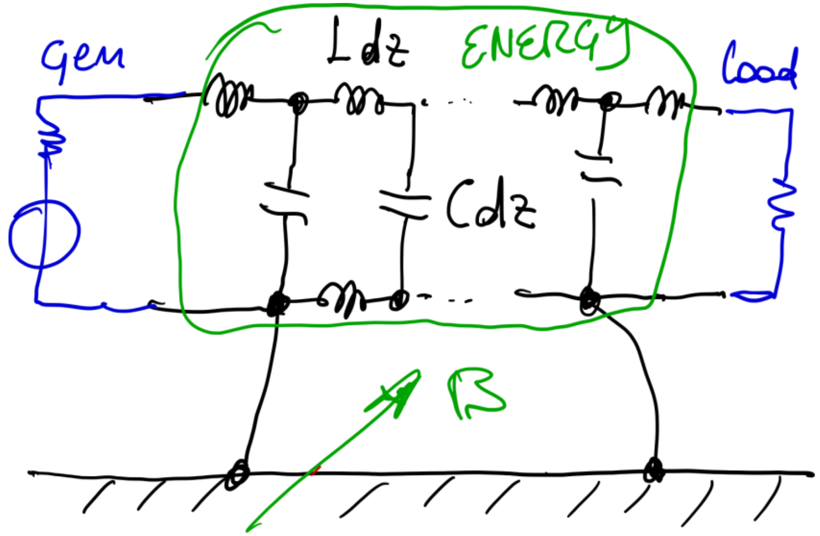
- Mumetal
  - Various compositions, about Ni 77%, Fe 16%, Cu 5%, Cr 2%
  - Ductile/malleable
- Permalloy
  - Ni 80%, Fe 20%,
- $\mu_r = 10^5$
- Require annealing
- Suffer shocks/acceleration

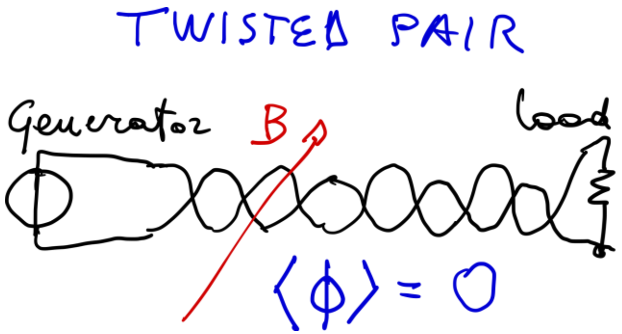
Superconductors are perfect (and impractical) electric and magnetic shields (Meissner effect)

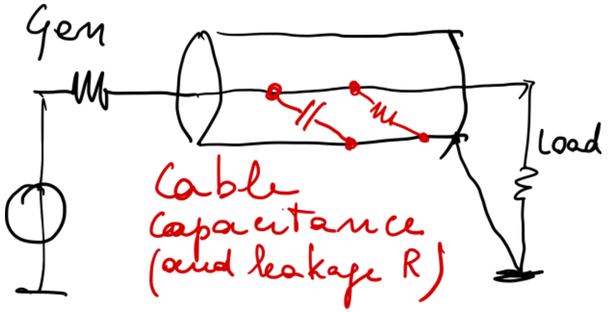
Guarding and shielding



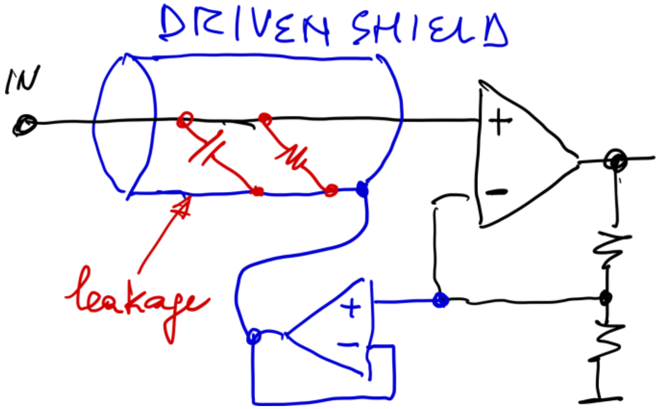
RF in cables & twisted pairs propagates as a field
cutoff frequency f_c = 2…10 kHz
ground loops allowed (far) beyond f_c

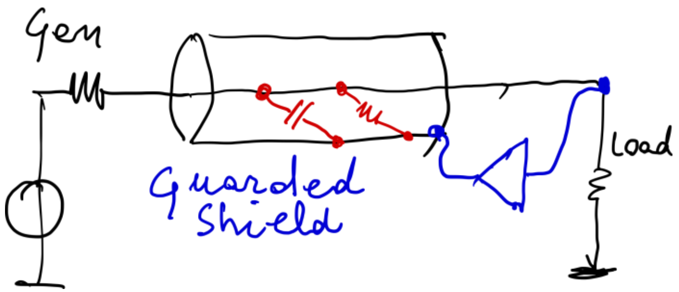


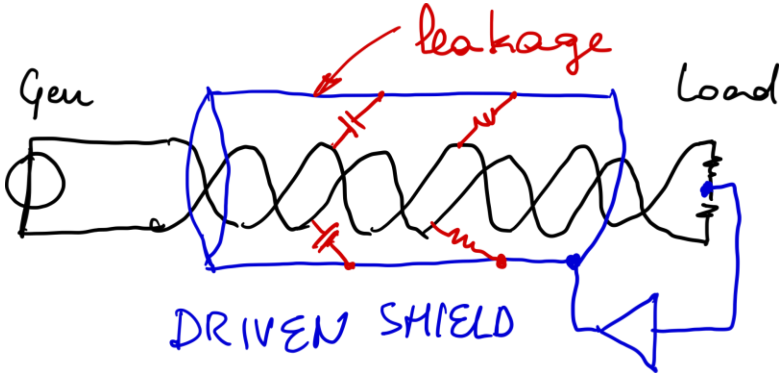




Featured readings
H. W. Ott, Electromagnetic Compatibility Engineering, Wiley 2009, ISBN 978-0-470-18930-6
C. R. Paul, Introduction to Electromagnetic Compatibility, Wiley 2006, ISBN 978-0-471-75500-5

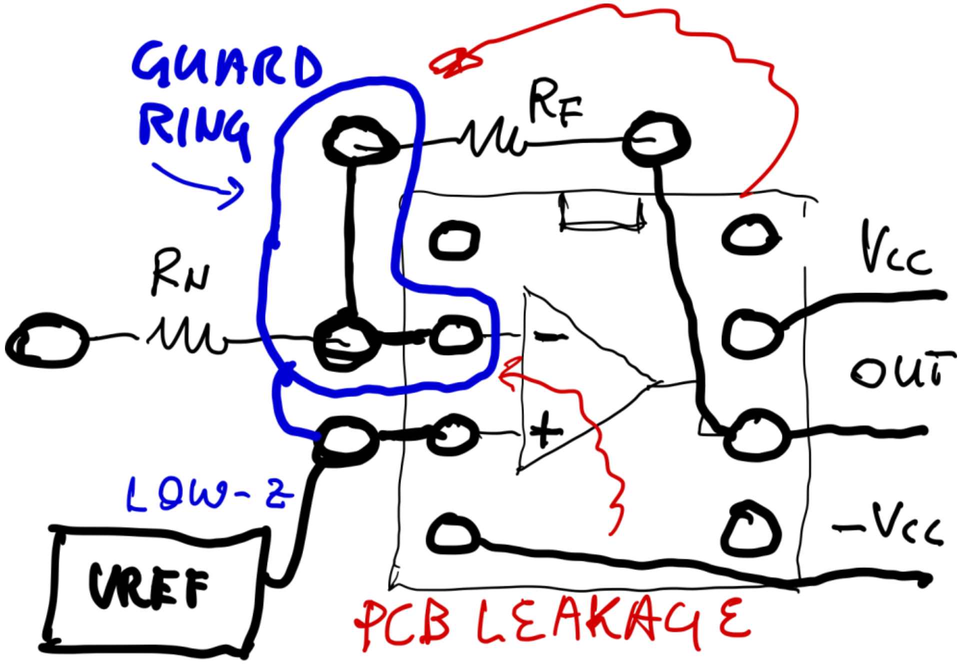




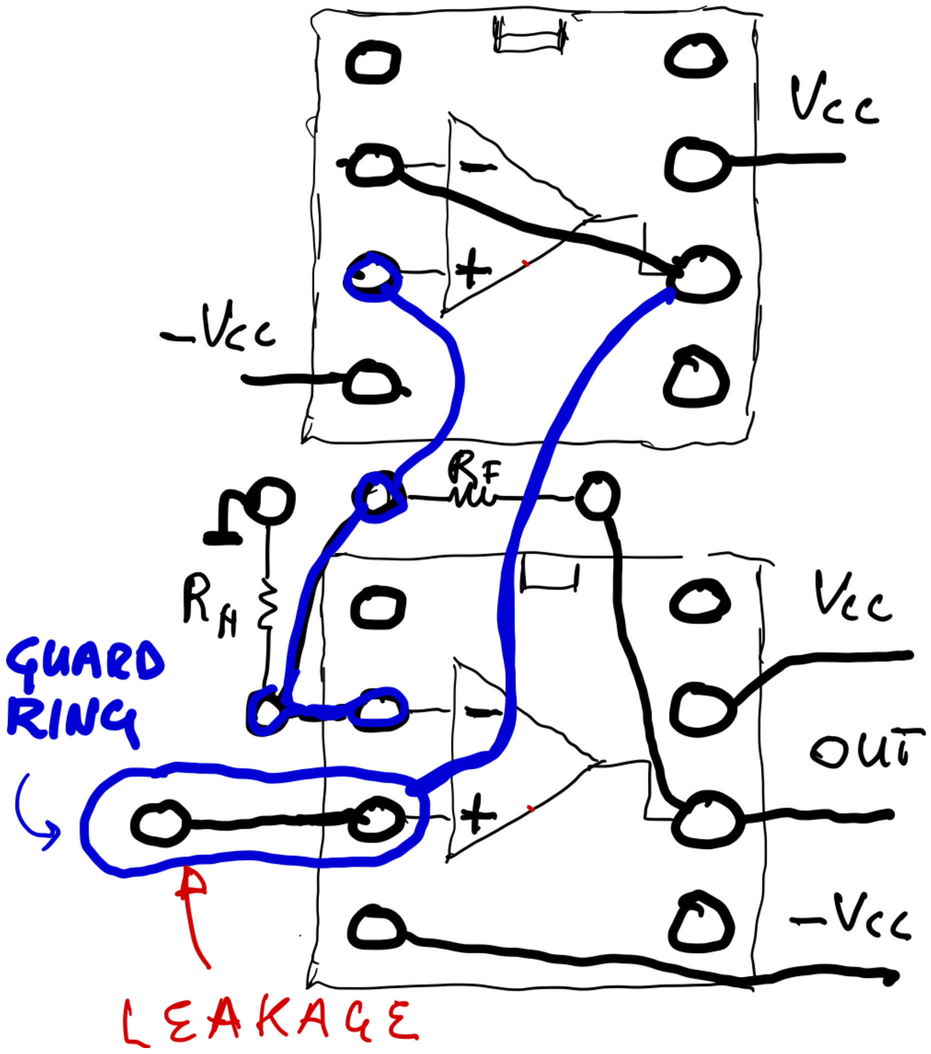


Printed circuit boards

Inverting amplifier



Non inverting amplifier



Standard operational amplifier,
8-pin DIL package, top view

Lecture 2 ends here

Lecture 3

Scientific Instruments & Oscillators

Spring 2022

Lectures for PhD Students and Young Scientists

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Contents

- Noise in RF/microwave devices (cont)
- Photodetectors
- Analog-to-digital and digital-to-analog conversion

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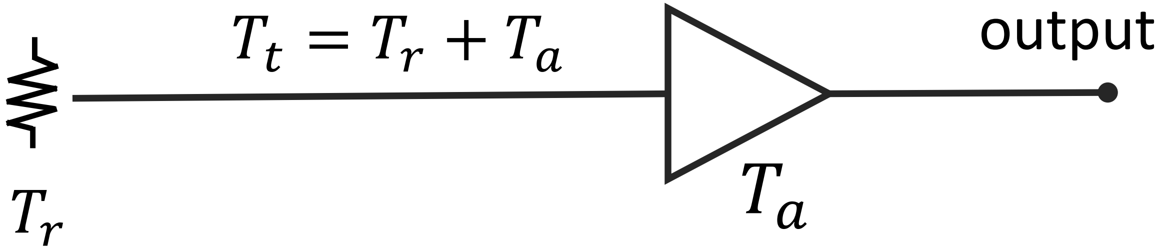
home page http://rubiola.org

Equivalent noise temperature

Thermal noise

$$S(\nu) = \frac{h\nu}{e^{h\nu/kT}} \qquad S(\nu) = kT \quad \text{constant, for } h\nu \ll kT$$



Ta is the equivalent noise temperature of the amplifier defined in specified conditions (physical temperature and input resistance)

Equivalent temperature T_a defined by $S(\nu) = k(T_a + T_r)$

- Warning: the noise temperature a radio-engineering concept
- The physical nature of noise does not matter
- Often misleading in optics: the shot noise contributes to the equivalent temperature

Belongs to PM noise

Homework

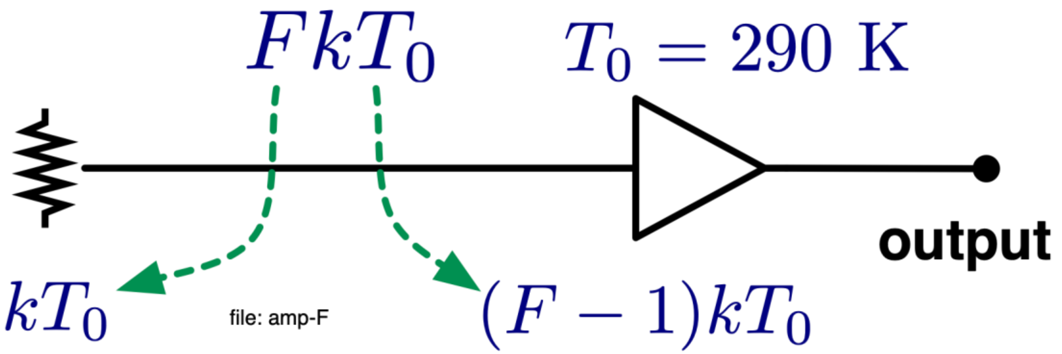
- Work out the noise temperature of the operational amplifier at $R_{\text{best}} = e_n/i_n$
- Calculate T_{eq} for the OP27 and the LT1028
- You should find almost the same T_{eq} , despite the fact that the noise of the two amplifier is so different.
- Can you figure out why?

Noise factor and noise figure

Noise factor $F = \frac{\mathrm{SNR(out)}}{\mathrm{SNR(in)}}$ general definition

$$F = \frac{\mathrm{SNR_{out}}}{\mathrm{SNR_{in}}}$$

Noise Figure
NF = 10 $\log_{10}$(F)



$$kT_0 = 4 \times 10^{-21}\mathrm{J}$$

–174 dBm/Hz

290 K (17 ℃) is a convenient round number

Assume that the whole circuit is at the reference temperature $T_0 = 290$ K (17 ℃)

The total noise referred to the amplifier input is FkT_0

amplifiers and RF/µw devices

$$FkT_0 = kT_e = k(T_a + T_0), \qquad T_0 = 290\ \mathrm{K}$$

$$F = \frac{(T_a + T_0)}{T_0} \quad \text{and} \quad T_a = (F-1)T_0$$

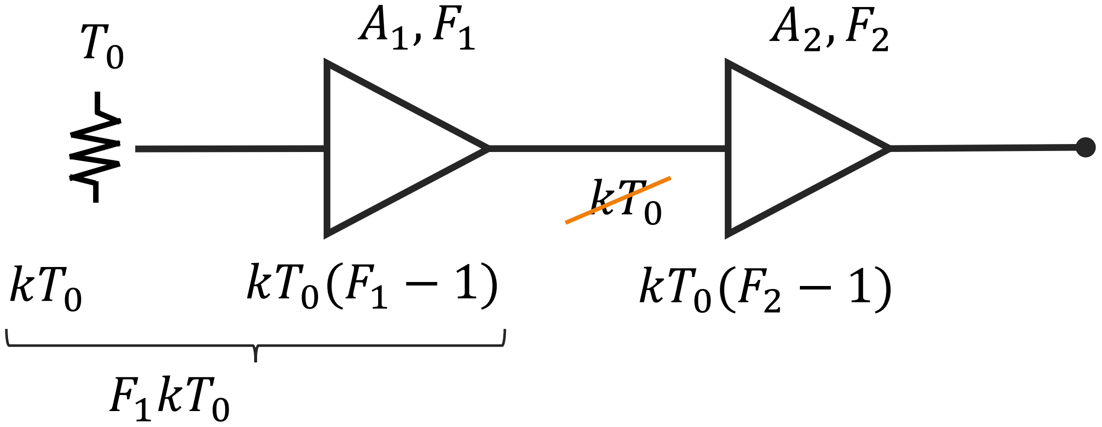
Warning: the noise figure is a radio-engineering concept, may be ***misleading*** in ***optics***

Belongs to PM noise

The Friis formula

A = voltage gain

A^2 = power gain



$$N = kT_0 + (F_1 - 1)kT_0 + \frac{(F_2 - 1)kT_0}{A_1^2}$$

$$F = F_1 + \frac{(F_2 - 1)}{A_1^2} + \frac{(F_3 - 1)}{A_1^2 A_2^2} + \cdots$$

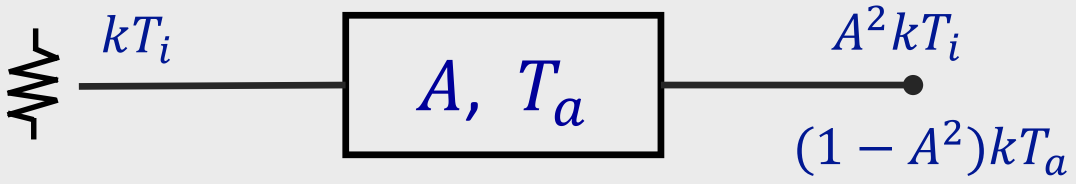
main maybe negligible

Caveat

- Impedance matching not included
- Three different conditions
  - Max power transfer
  - Lowest noise
  - Highest SNR

H. T. Friis, Noise Figures of Radio Receivers, Proc IRE 32(7) p419-422, July 1944

POI – Thermal noise of a dissipative device



$$S(f) = A^2kT_i + (1-A^2)kT_a$$

Describes noise in

- Cables
- Antennas
- Propagation in lossy medium

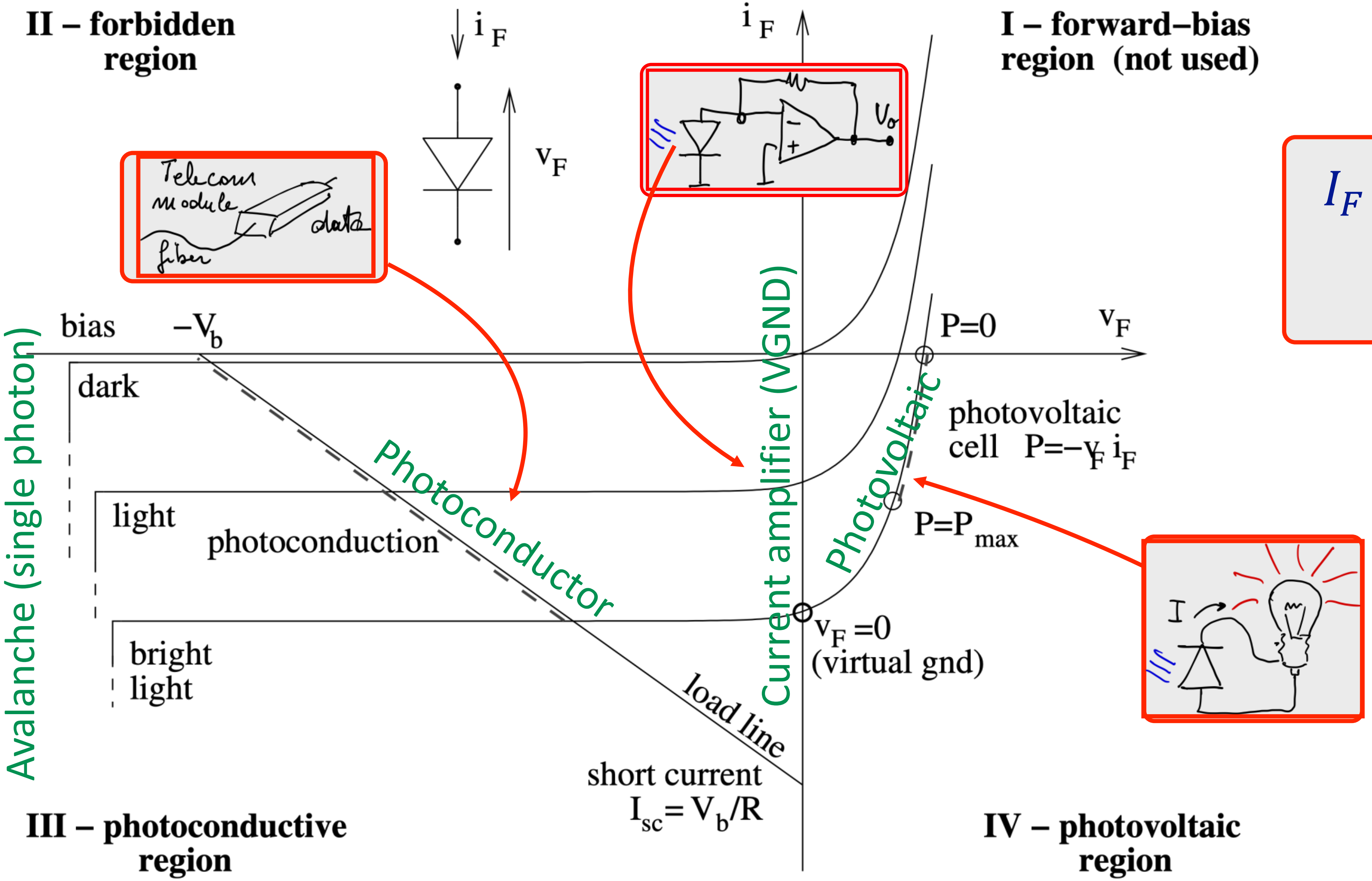
Arno A. Penzias and Robert W. Wilson (Nobel in Physics, 1978) knew about noise temperature when they measured the background cosmic radiation

- Noise contribution of the input resistor
- The attenuator makes no difference between “noise” and “signal”
- The input signal is “amplified” by a factor $A^2 < 1$
- Noise contribution of the attenuator
- At uniform temperature T, the sum of the contributions must be kT
- The input contributes A^2kT
- The attenuator contributes the complement $(1-A^2)kT$
- The factors A^2 and $1-A^2$ do not depend on temperature

Featured readings
A. A. Penzias, R. W. Wilson, A Measurement of Excess Antenna Temperature at 4080 Mc/s, Astrophys J Lett.142(1), p.419-421, 1965
J. D. Kraus, Antennas 2ed, McGraw Hill 1997, ISBN 0-07-035422-7
(The proof is found in Kraus, 1st ed., 1966, Sec.7-2b)

Belongs to PM noise

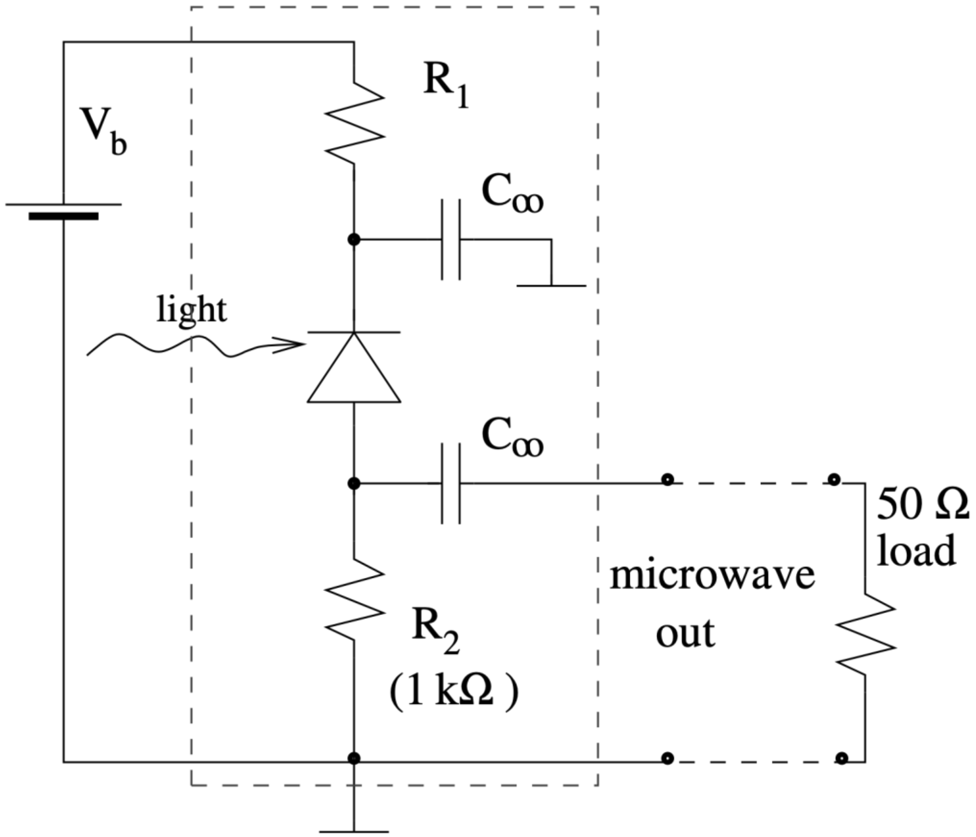
Photodiode

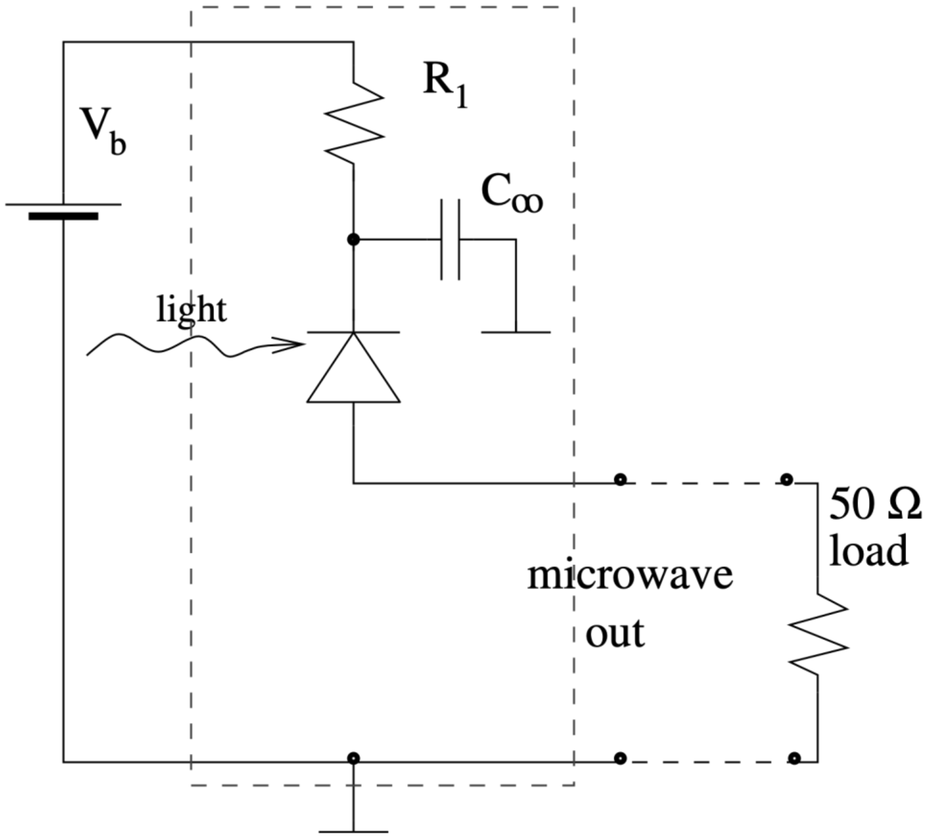


$$I_F = I_s(e^{V_F/V_T} - 1) - I_P$$

$$V_T = kT/e$$

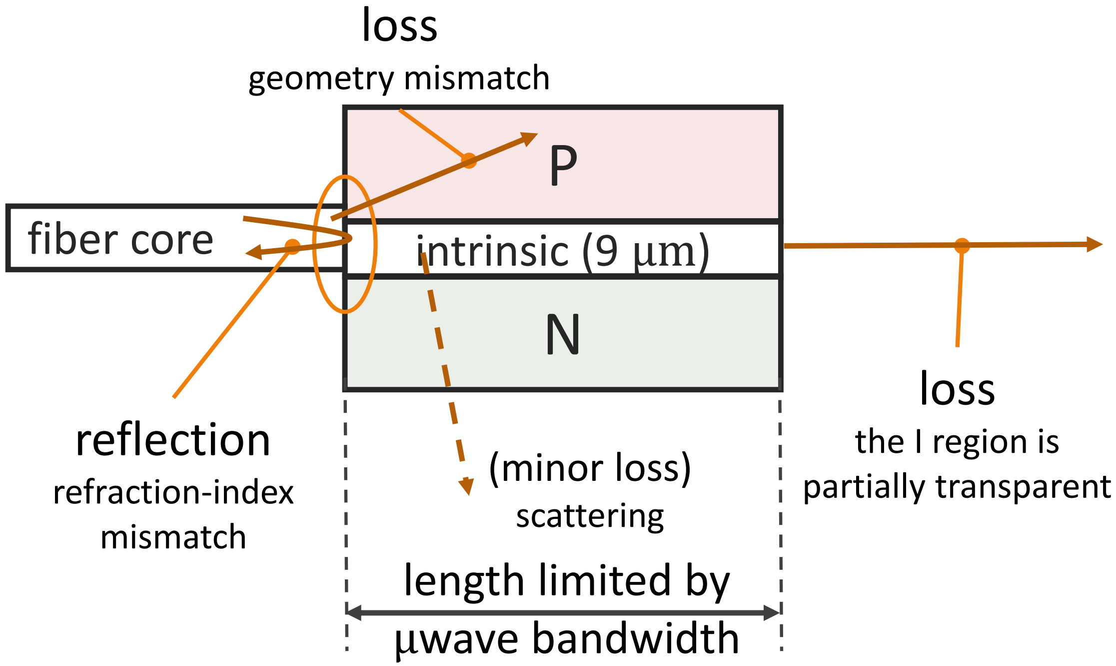
Fast photodiodes





Quantum efficiency and noise

High-speed PIN photodetector



ϕ photons / s

$\eta\phi$ detected, $(1-\eta)\phi$ lost

$$\phi = \frac{P}{h\nu}$$

Shot Noise

$$\bar{I} = \eta e \phi = \frac{\eta e}{h\nu}\overline{P}_\lambda$$ Current

$$S_I(f) = 2e\frac{\eta e}{h\nu}\overline{P}_\lambda \mathrm{A}^2/\mathrm{Hz}$$ Current PSD

$$N_s = 2e\frac{\eta e}{h\nu}R\overline{P}_\lambda \mathrm{W}/\mathrm{Hz}$$ Output-power PSD

Shot noise and thermal noise of the resistor

$$2qIR = kT$$

$$I = \frac{kT}{2eR} \quad N_s = N_t$$ threshold, shot = thermal

$$T_{\mathrm{eq}} = \frac{2eI}{k}$$ equivalent temperature !!!

Photodetector signal and noise

Photodetector signal

$$I = \rho P = \frac{\eta e}{h\nu} P \quad [\mathrm{A}]$$

Noise: shot, dark current, thermal (load), $I_s \equiv I$

$$S_I = 2e(I_s + I_d) + 4kT/R[\mathrm{A}^2/\mathrm{Hz}]$$

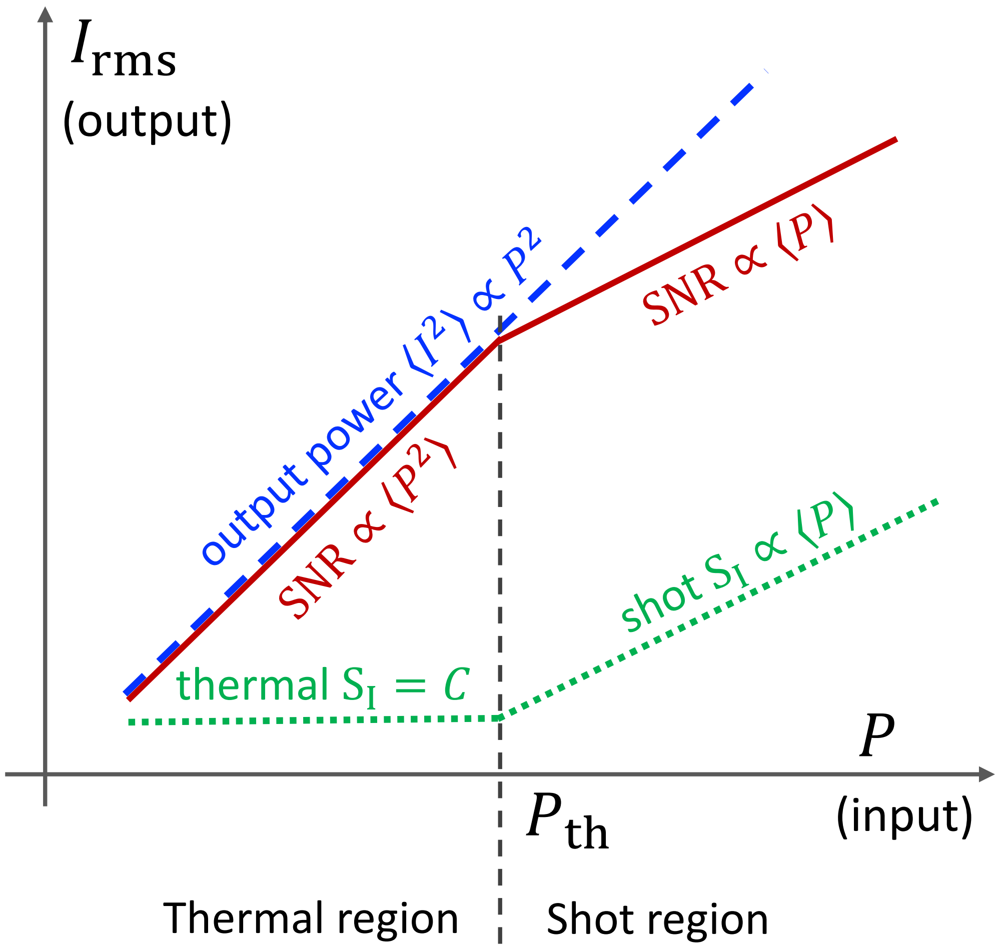
Shot is dominant at high P

$$S_I = 2eI = \frac{2e^2\eta P}{h\nu}$$

Threshold $S_{\mathrm{sh}} = S_{\mathrm{th}}$,

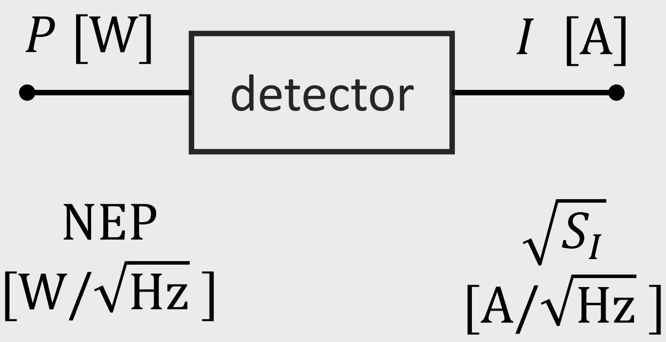
$$P_{\mathrm{th}} = 2\,\frac{h\nu}{e^2\eta}\frac{kT}{R}$$

Thumb rule: P ≈ 1 mW, 1.5 µm, 50 Ω, 300 K



Noise Equivalent Power (NEP)

responsivity



P [W]

detector

I [A]

NEP $[\mathrm{W}/\sqrt{\mathrm{Hz}}\,]$

$\sqrt{S_I}$ $[\mathrm{A}/\sqrt{\mathrm{Hz}}\,]$

The output can be I, V, or any other quantity
(including a number at the output of an ADC)

Don't mistake
optical power P at the input
signal power σ_s^2 at the output
noise power σ_n^2 at the output

- Radiometric concept
- Applies to quantum detectors, bolometers, and any other radiation detector

The NEP is the input power in that gives

$\mathrm{SNR} = 1$ (Signal-to-Noise Ratio) in 1 Hz bandwidth

$$\mathrm{NEP}^2 = \frac{P^2}{\Delta f} \quad \text{at} \quad \sigma_n^2 = \sigma_s^2$$

Example: Photodiode

$$\sigma_s^2 = I^2 = \left(\frac{e\eta P}{h\nu}\right)^2 \qquad \sigma_n^2 = S_I(f)\Delta f$$

Featured reading: S. Leclerq, Discussion about Noise Equivalent Power and its use for photon noise calculation, March 2007
Available: http://www.iram.fr/~leclercq/Reports/About_NEP_photon_noise.pdf (retrieved April 2020)
Also: P. L. Richards, Bolometers for infrared and millimeter waves, J Appl Phys 76(1) p.1-25, 1 July 1994

NEP in photodetectors

Low power, thermal region

$$S_I = \frac{4kT}{R}$$

(signal)^2 = (noise)^2 in Δf

$$\left(\frac{e\eta P}{h\nu}\right)^2 = \frac{4kT}{R}\Delta f$$

$$\frac{P^2}{\Delta f} = \frac{h^2\nu^2}{e^2\eta^2}\frac{4kT}{R}$$

$$\text{NEP} = 2\frac{h\nu}{e\eta}\sqrt{kT/R}$$

Thumb rule:

NEP = 1.8x10^{-11} W/√Hz,

1.5μm, 50Ω, 300K, η=0.8

High power, shot region

$$S_I = 2\frac{e^2\eta P}{h\nu}$$

(signal)^2 = (noise)^2 in Δf

$$\left(\frac{e\eta P}{h\nu}\right)^2 = 2\frac{e^2\eta P}{h\nu}\Delta f$$

do not simplify *P*

$$\frac{P^2}{\Delta f} = \frac{h^2\nu^2}{e^2\eta^2}2\frac{e^2\eta P}{h\nu}$$

$$\text{NEP} = \sqrt{2\frac{h\nu}{\eta}P}$$

Thumb rule:

NEP = 1.8x10^{-11} W/√Hz,

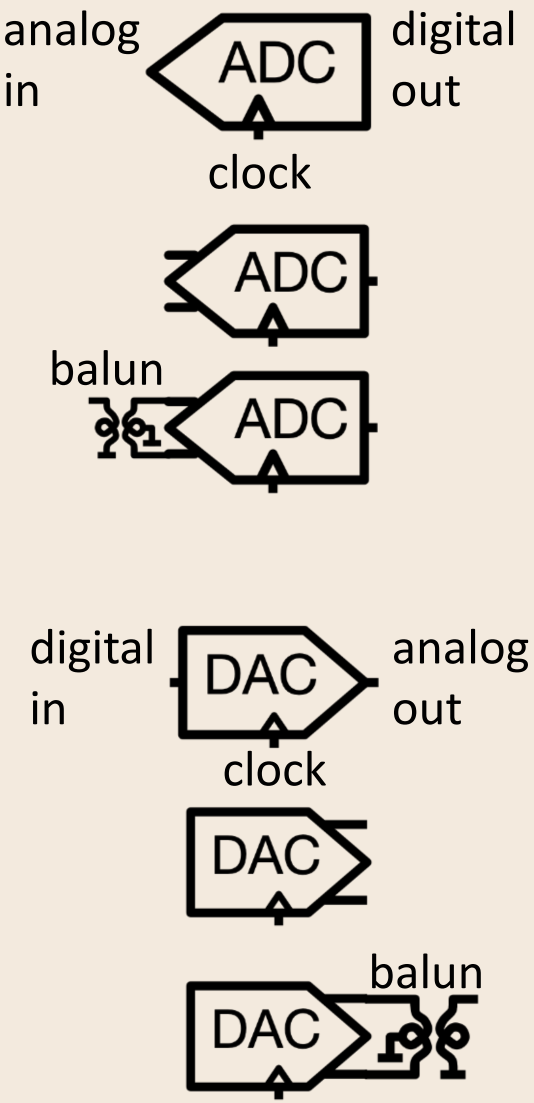
1.5μm, η=0.8, P=1mW

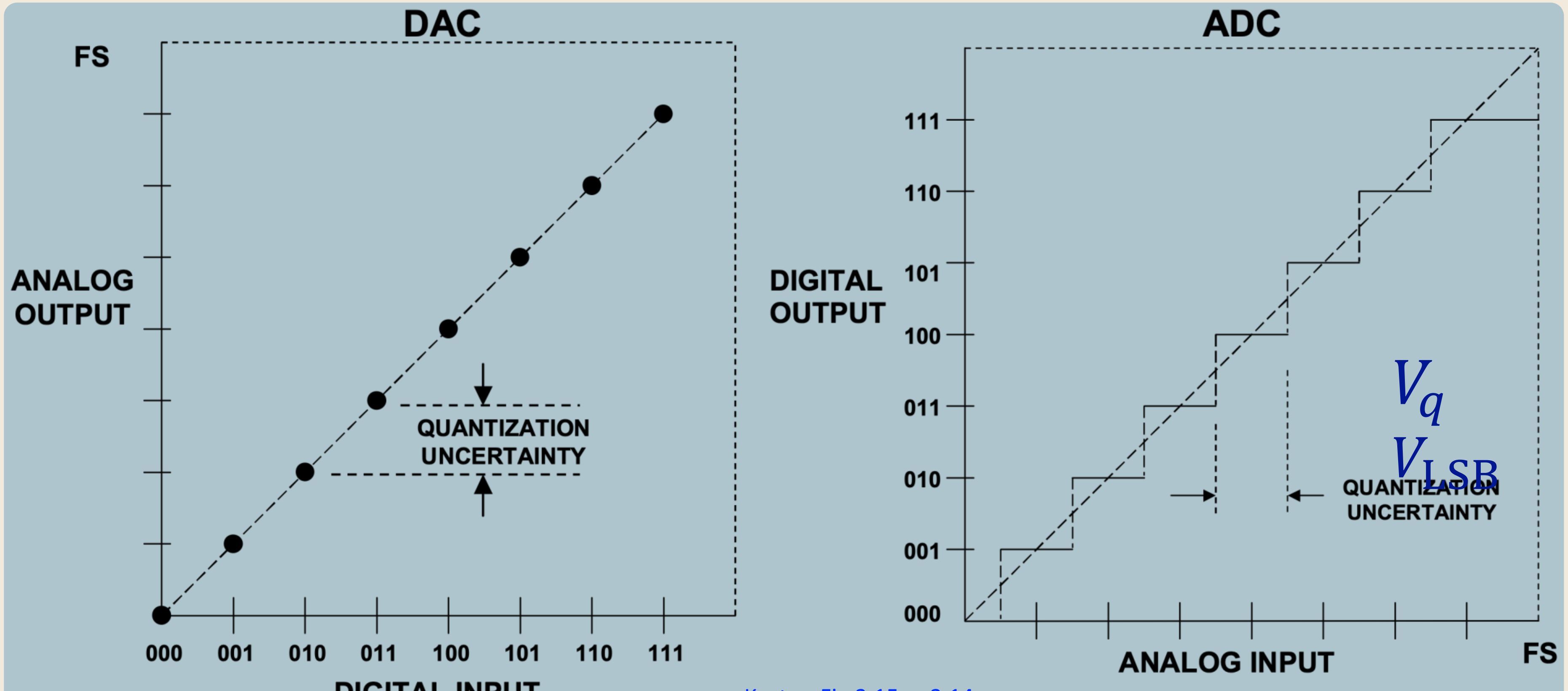
Analog-to-Digital Conversion

Excerpt from
06 Digital

ADCs and DACs

Skip





V_q

V_{LSB}

Kester, Fig.2.15, p.2.14

New, put to PM noise

Featured reading: Kester W (ed), *Analog-Digital Conversion*,
Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

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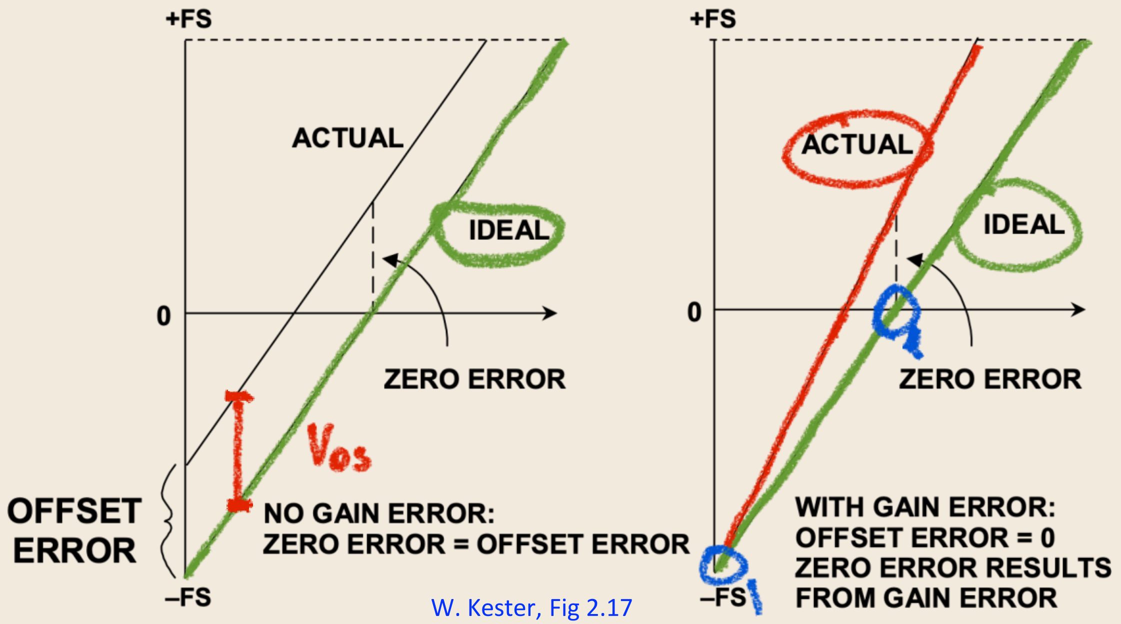
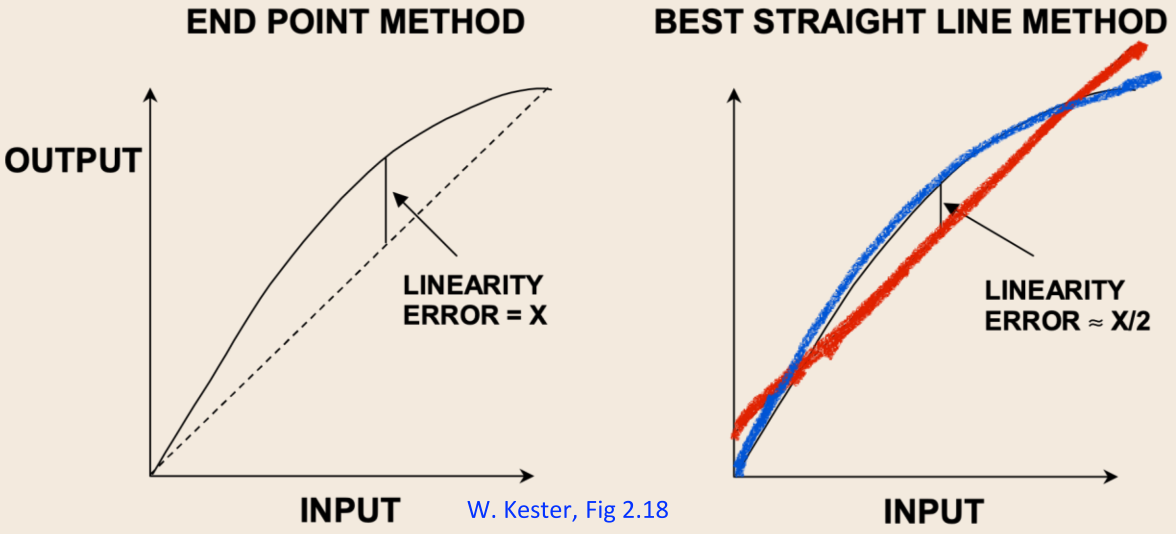
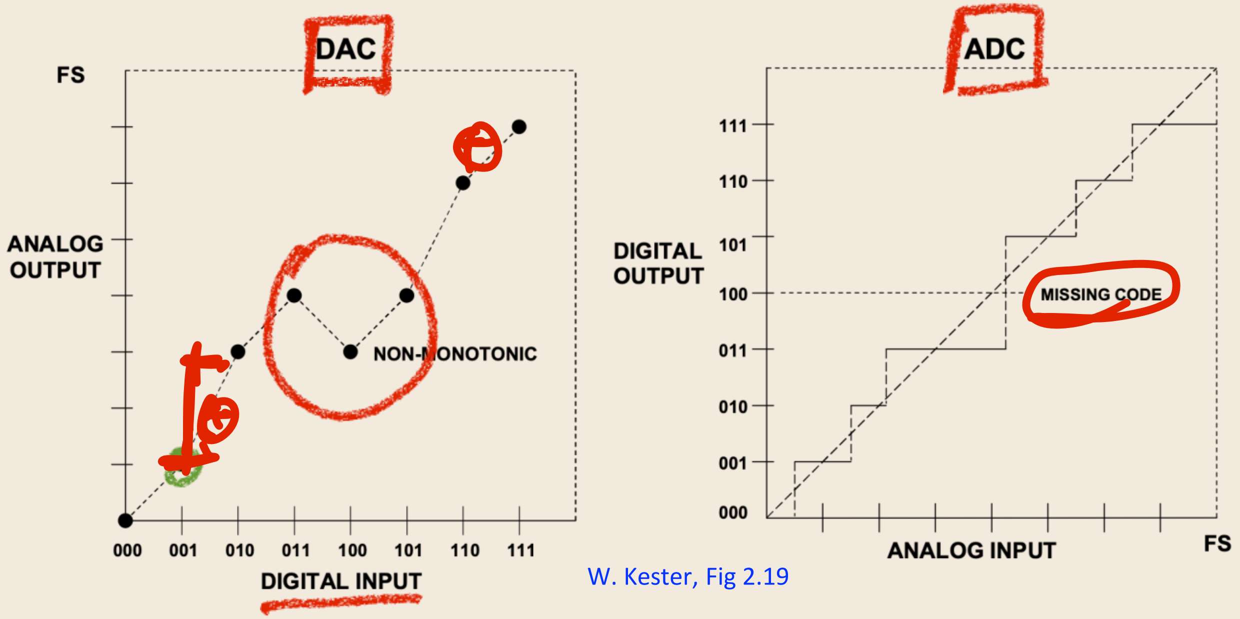
Basic Non-Idealities



W. Kester, Fig 2.17



W. Kester, Fig 2.18



W. Kester, Fig 2.19

Skip

Figures: Kester W (ed), *Analog-Digital Conversion*, Fig.2.15, p.2.14, Analog Devices 2004, ISBN 0-916550-27-3

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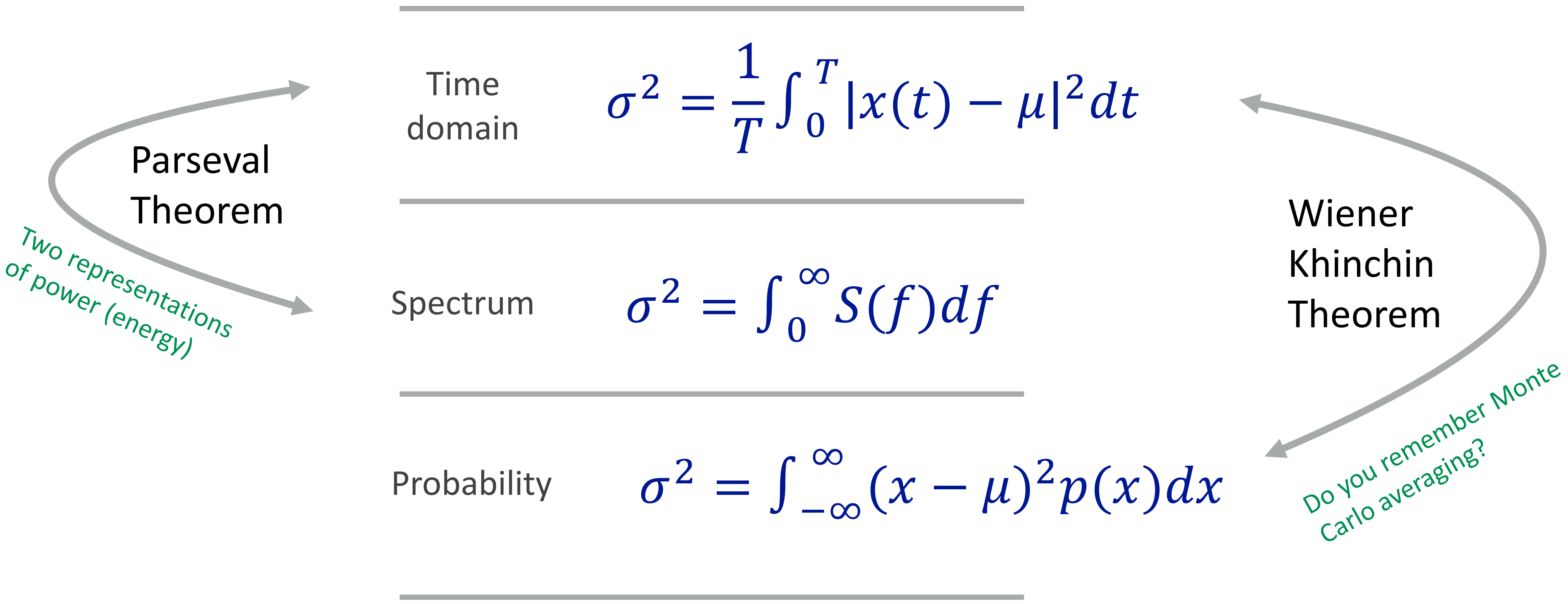
Variance (Signal Power)

The variance σ^2 (power P) of a signal can be evaluated in
(i) time domain, (ii) frequency domain or (iii) probability, and the result is the same



Time domain

$$\sigma^2 = \frac{1}{T}\int_0^T |x(t) - \mu|^2 dt$$

Spectrum

$$\sigma^2 = \int_0^\infty S(f) df$$

Probability

$$\sigma^2 = \int_{-\infty}^\infty (x - \mu)^2 p(x) dx$$

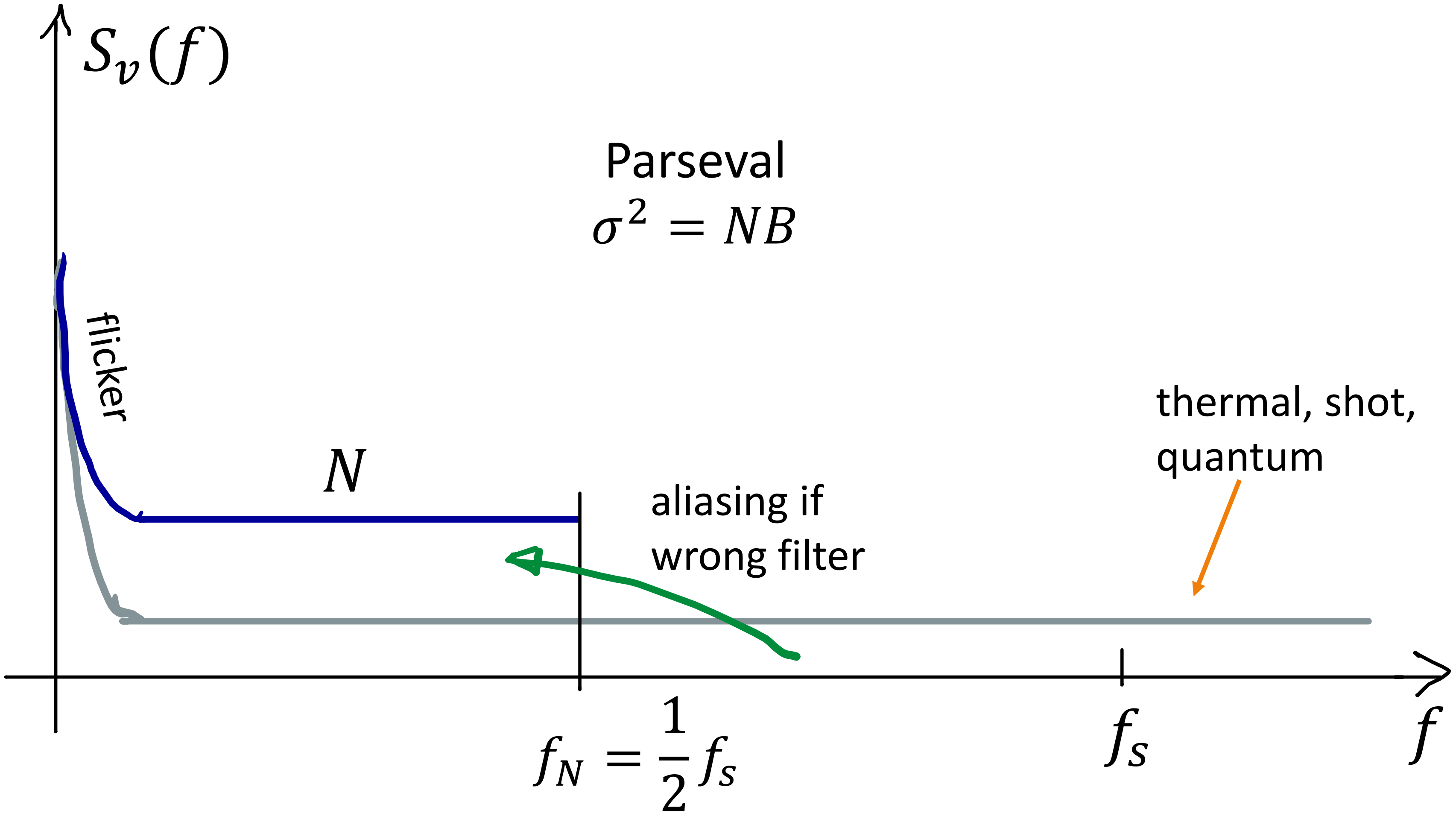
Parseval Theorem

Two representations of power (energy)

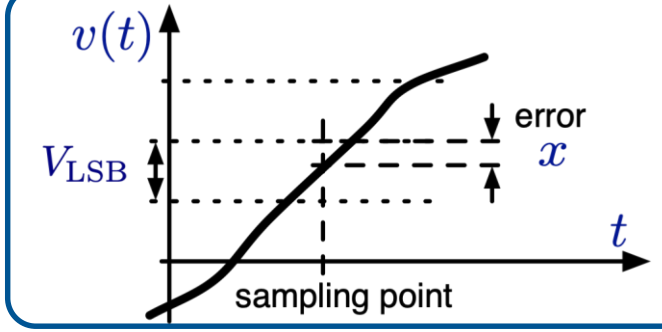
Wiener Khinchin Theorem

Do you remember Monte Carlo averaging?

Ultimate Limits

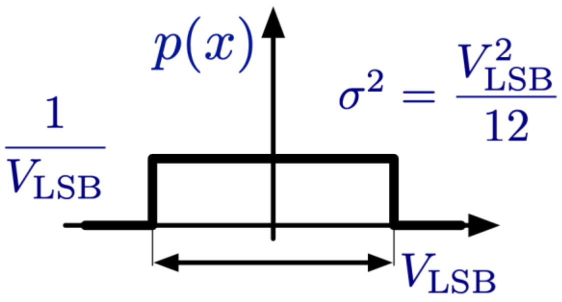


Quantization Noise



Analog-to-digital conversion introduces a quantization error x $[-VLSB/2 \leq x \leq +VLSB/2]$

$$n\text{-bit conversion: } V_{\rm LSB} = \frac{V_{\rm FSR}}{2^n}$$



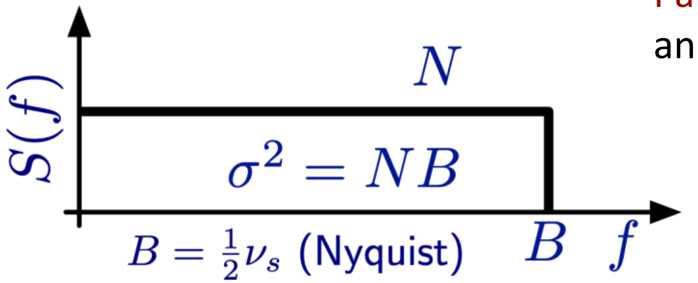
Wiener–Khintchine theorem: in ergodic systems, interchange time & ensemble

The noise can be calculated with statistics

$$\sigma^2 = \frac{V_{\rm FSR}^2}{12 \times 2^{2n}} \qquad \mathrm{V}^2$$

1/12 –> –10.8 dB

2^{2n} –> 6 dB/bit



Parseval theorem: Energy (power) calculated in time and in frequency is the same

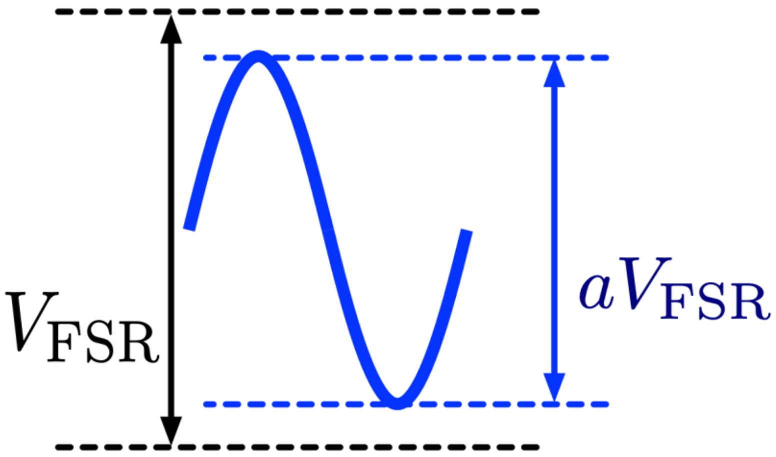
$$N = \frac{V_{\rm FSR}^2}{6 \times 2^{2n}\, \nu_s} \qquad \mathrm{V}^2/\mathrm{Hz}$$

Sampling frequency is f_s, not ν_s

Featured reading: W. R. Bennett, Spectra of quantized signals, Bell System Tech J. 27(4), July 1948

Widrow B, Kollar I, *Quantization Noise*, Cambridge 2008, ISBN 978-9-521-88671-0

Quantization & Sinusoidal Signals



Assume that the noise power is equally distributed between 0 and $B = f_s/2$

This is not true when signal and clock are highly coherent (Widrow-Kollar, Appendix G)

Provisionally, take uniform distribution

$$P_0 = \frac{V_{pp}^2}{8} = \frac{a^2 V_{\mathrm{FSR}}^2}{8}$$

$$P = \frac{V_{pp}^2}{8} = \frac{a^2 V_{\mathrm{FSR}}^2}{8}$$ Signal power

$$\sigma^2 = \frac{V_{\mathrm{LSB}}^2}{12}$$ Noise power

$$\frac{V_{\mathrm{FSR}}}{V_{\mathrm{LSB}}} = 2^n$$ Quantization

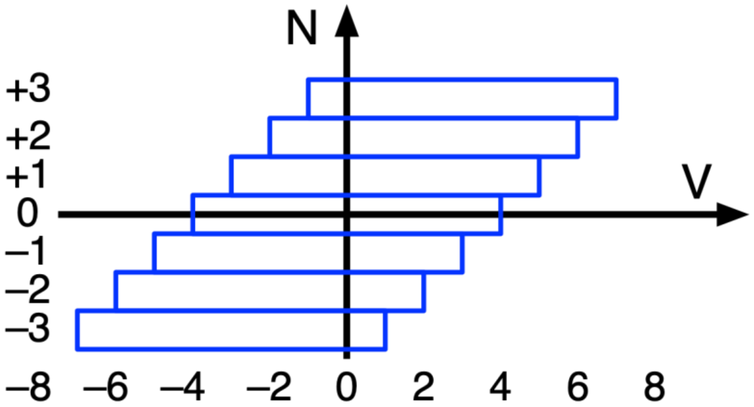
$$\mathrm{SNR} = \frac{P}{\sigma^2} = \frac{3}{2} 2^{2n} a^2$$

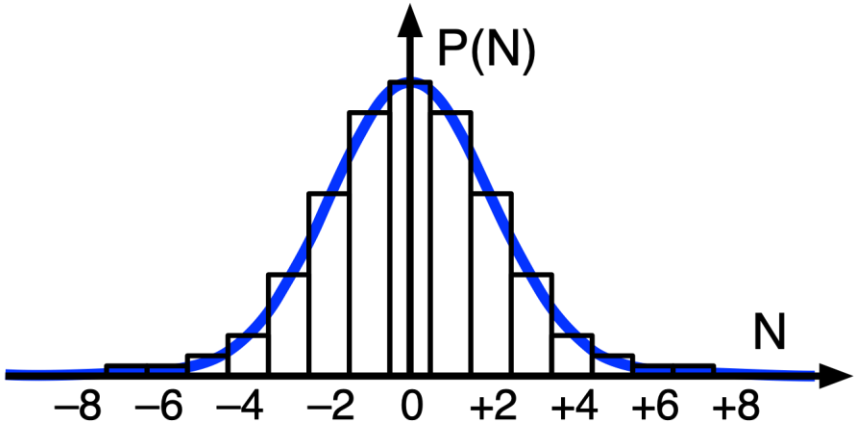
Often seen as $1.76 + 6.02 \log_{10}(n)$ dB (with $a = 1$)

$$\frac{N}{P} = \frac{V_{\mathrm{FSR}}^2}{6 \times 2^{2n} f_s} \frac{8}{a^2 V_{\mathrm{FSR}}^2}$$

$$\frac{N}{P} = \frac{4}{3} \frac{1}{2^{2n} a^2 f_s}$$

Transition Noise



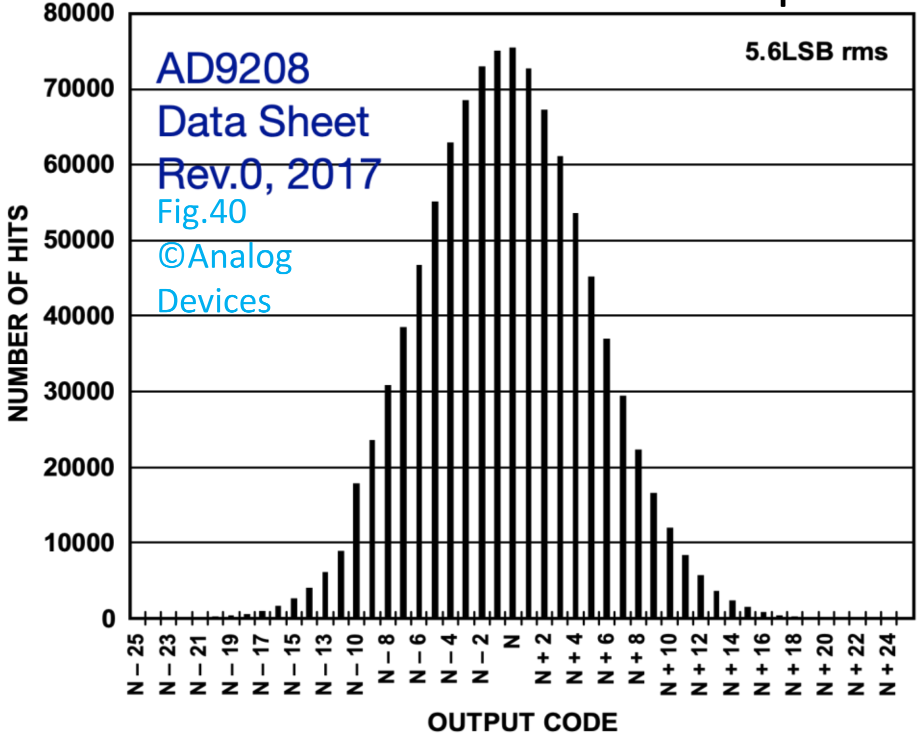


- Actual noise includes quantization, analog noise, and distortion

$$\sigma^2 = \sigma_q^2 + \sigma_a^2 + \sigma_d^2$$

- Random distribution of output N
- Metrology suggests to make σ_q^2 negligible
  - BUS bits are cheap
  - Analog precision complex / fundamental limits

Example



AD9208 Data Sheet Rev.0, 2017 Fig.40 ©Analog Devices

Resolution and Entropy

Entropy (information theory)

$$H = -\sum_{i=1}^{N} p_i \log_2(p_i) \quad \text{[bit]}$$

Example: 1024 equally probable values,
i.e. $p_1 = 1/1024$, $\log_2(p_i) = -10$, $N = 1024$

$$H = -1024\left[\frac{1}{1024} \times (-10)\right] = 10\text{bit}$$

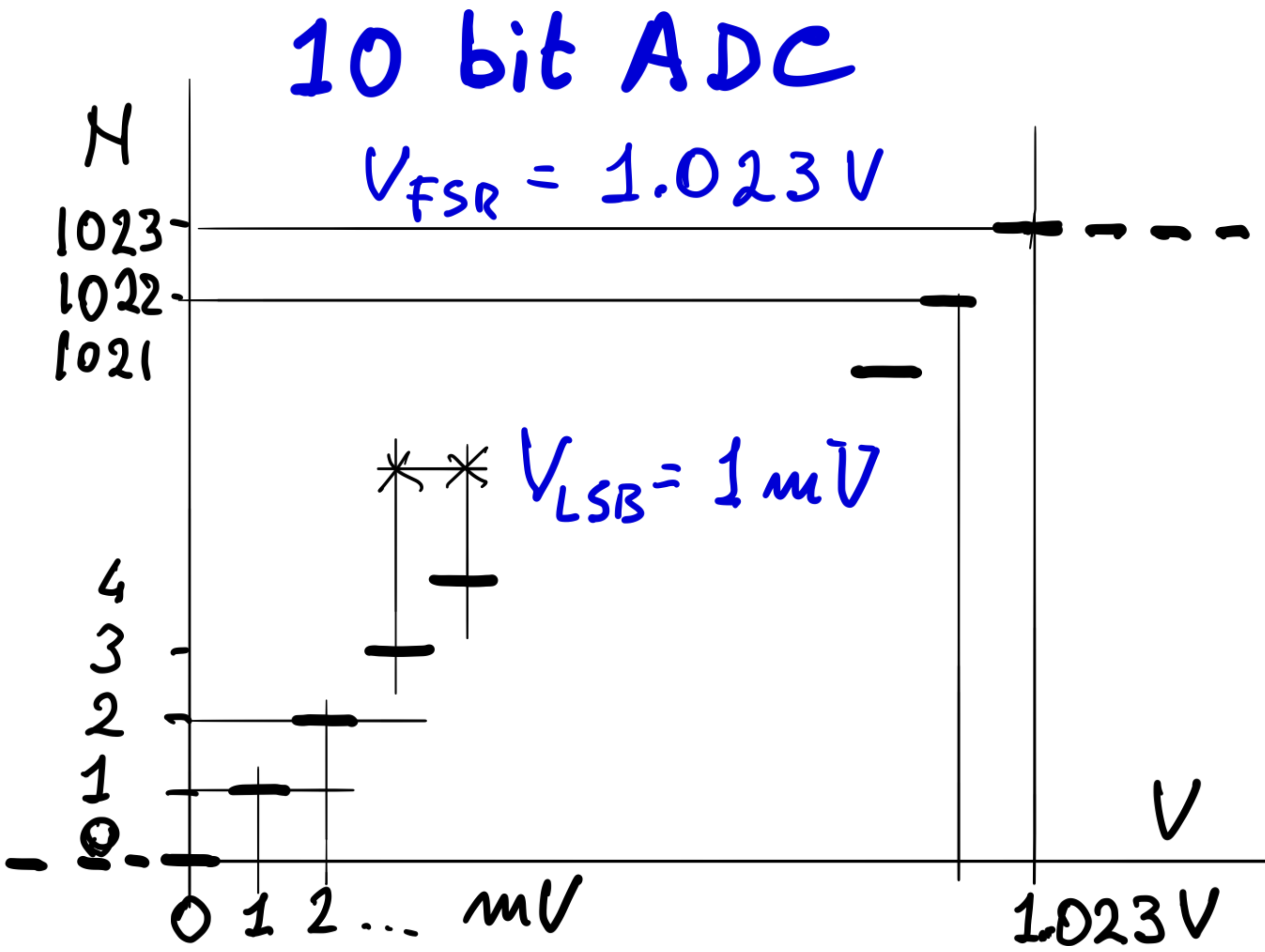
Non-uniform probability —> $H < H_{\max}$

Entropy in ADC

$$n = \log_2\left(1 + \frac{V_{\mathrm{FSR}}}{V_{\mathrm{LSB}}}\right)$$

The number *n* of bits is the same thing as *H* (assumes uniform quantization)

Unit	bit	nat	Hartley
Log base	2	e	10



Exercise

Calculate the entropy of a so-called “3 ½ digit” voltmeter

- Full scale 2 V, resolution 1 mV
- Actual readout –1.999 V … +1.999 V

Give the result in digit (!!!) and bit

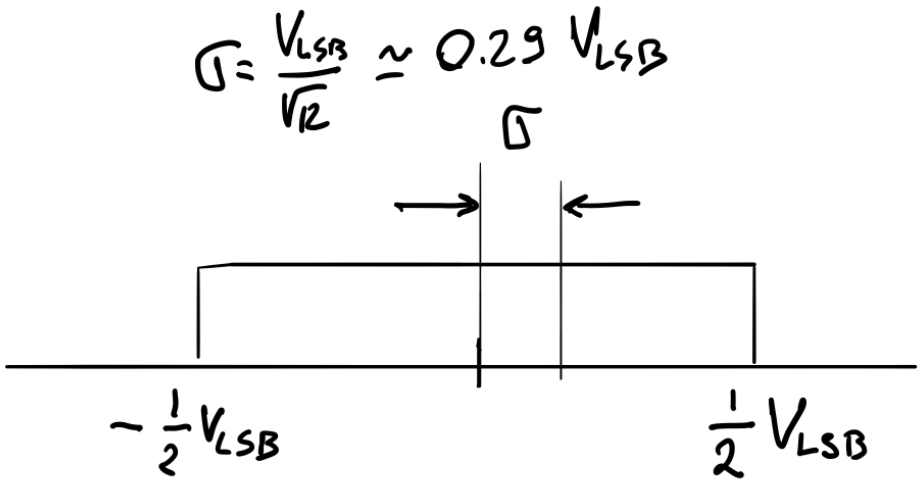
Entropy and Transition Noise

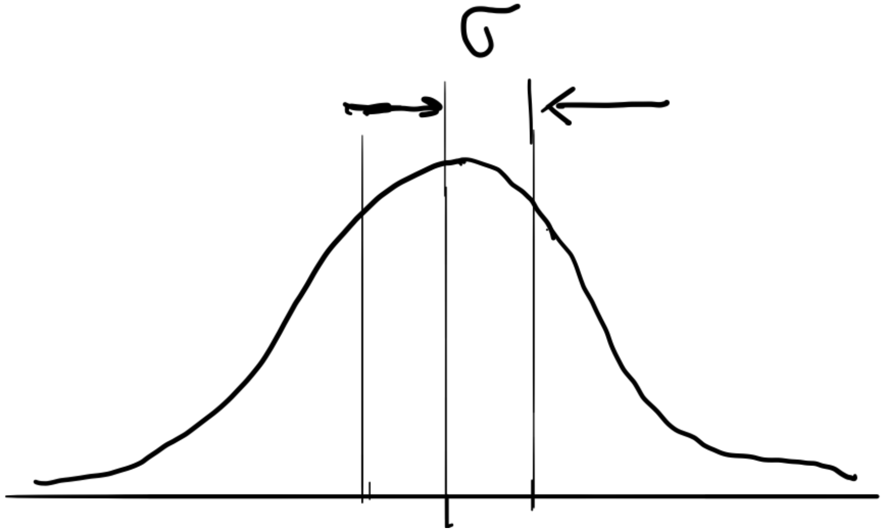
This is an approximation – Reality is way more complex, read Widrow & Kolar

$$H = \log_2\left(1 + \frac{V_{FSR}}{V_{LSB}}\right)$$

Replace $V_{LSB} \rightarrow \sqrt{12}\,\sigma$

$$H = \log_2\left(1 + \frac{V_{FSR}}{\sqrt{12}\sigma}\right)$$





Take this as a heuristic explanation.
This approximation is reasonably close to the exact result.

Widrow B, Kollar I, *Quantization Noise*, Cambridge 2008, ISBN 978-9-521-88671-0

Something Funny: The Maxwell's Demon

Skip

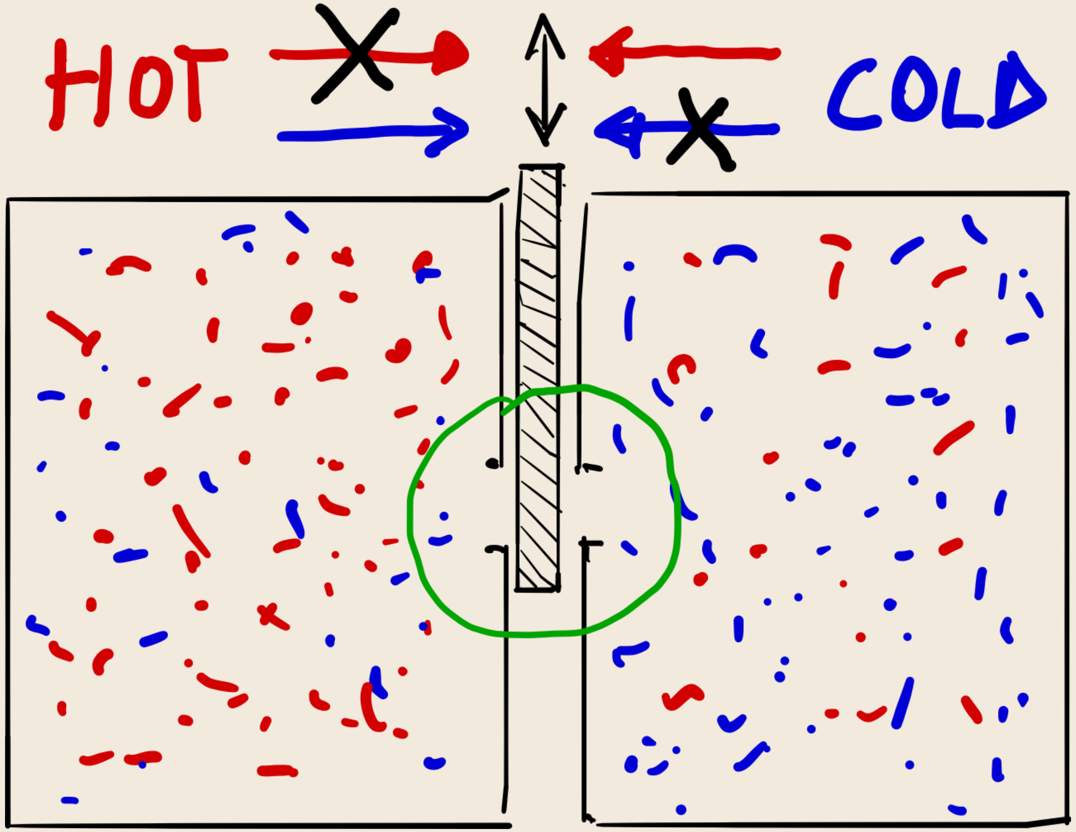
- Intriguing paradox
- Many scientists spent time and brainpower
- Theories: photon energy needed to probe the particles, etc.
- Ultimately, the ND shows the equivalence between thermodynamic entropy and information entropy
- W micro states with probability $1/W$

$$H = -\sum_{i=1}^{W} p\log(p) = \log(W)$$

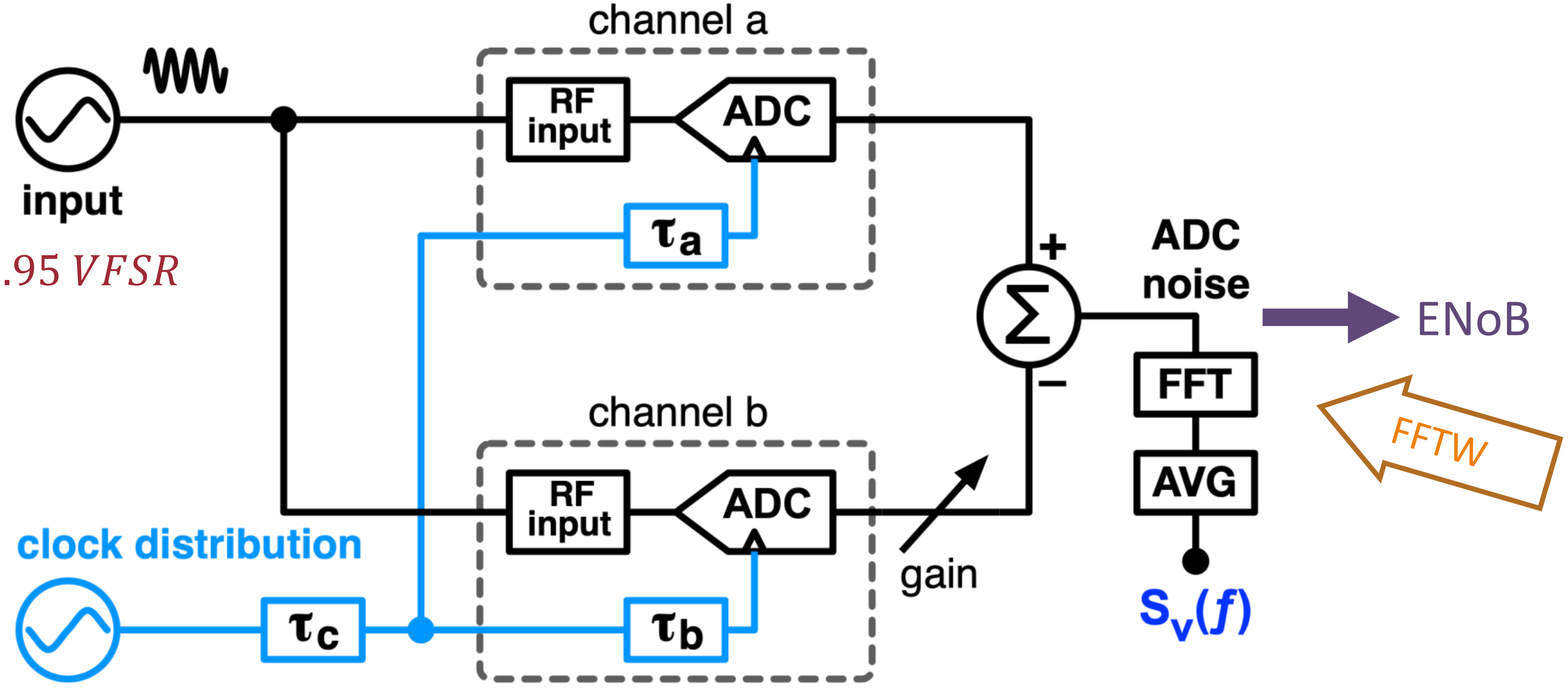
Units k per nat (nat is like bit, but in natural base)

$$H = k\log(W)$$

The demon
checks on the speed and allows
cold particles —> <— hot particles
The thermodynamical equilibrium is broken



Transition Noise Measurement



10 MHz

$V_{pp} \approx 0.95\, VFSR$

The differential clock jitter introduces additional noise due to the asymmetry between AM and PM

At 10 MHz input, the effect of ≈100 fs jitter does not show up

Effective No of Bits (ENoB)

Start from

$$\mathrm{SNR} = \frac{3}{2} 2^{2n}$$

Holds for full-scale-range sinusoidal signal

Replace

$$\mathrm{SNR} \to \mathrm{SINAD}$$

$$n \to \mathrm{ENoB}$$

SIgnal-to-Noise And Distortion ratio (quantization, analog noise and distortion)

$$\mathrm{SINAD} = \frac{P}{\sigma_q^2 + \sigma_a^2 + \sigma_d^2}$$

Usually expressed in bits, i.e., $\log_2(.)$

Get

$$\mathrm{SINAD} = \frac{3}{2} 2^{2\,\mathrm{ENOB}}$$

Usually written as

$$\mathrm{ENOB} = \frac{(\mathrm{SINAD})_{\mathrm{dB}} - 1.76}{6.02}$$

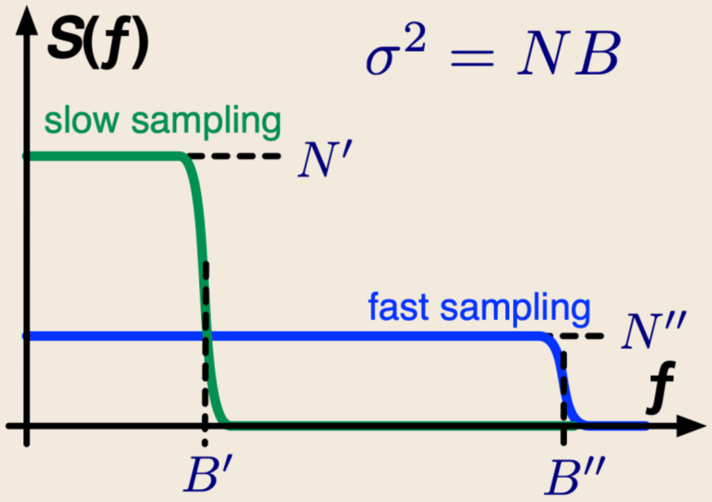
Lecture 3 ends here

Supplemental Material

Digital Filter and Decimation

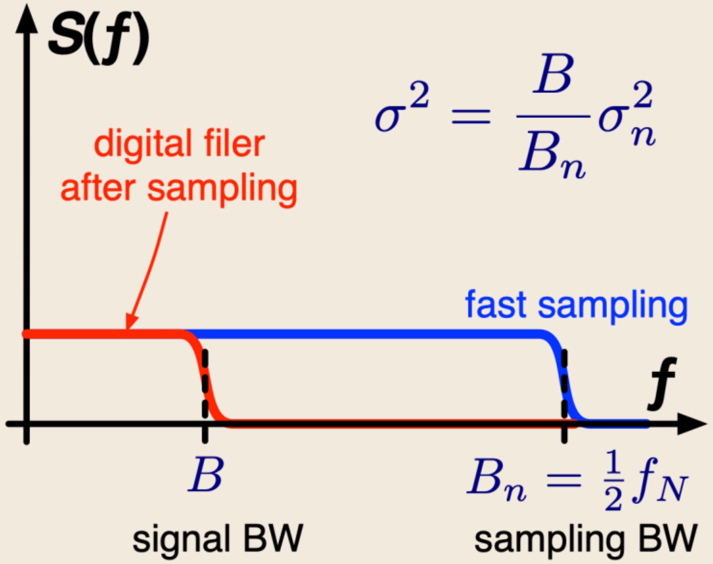
Noise, Sampling, and the
Parseval theorem

$$\sigma^2 = \frac{V_{\text{FSR}}^2}{12 \times 2^{2n}}$$



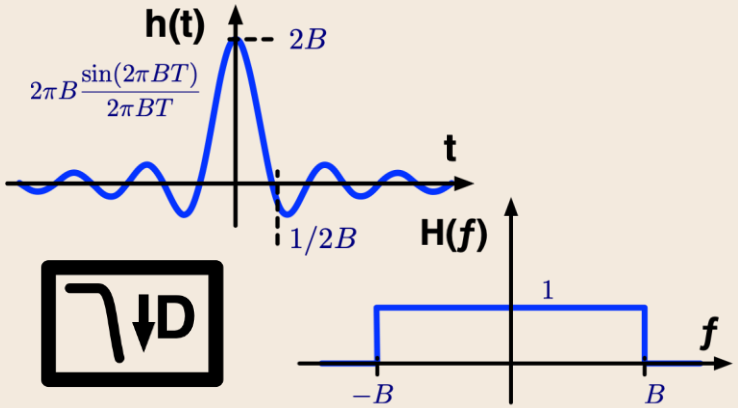
$$\sigma^2 = NB$$

Solution:
Fast sampling, and filter



$$\sigma^2 = \frac{B}{B_n} \sigma_n^2$$

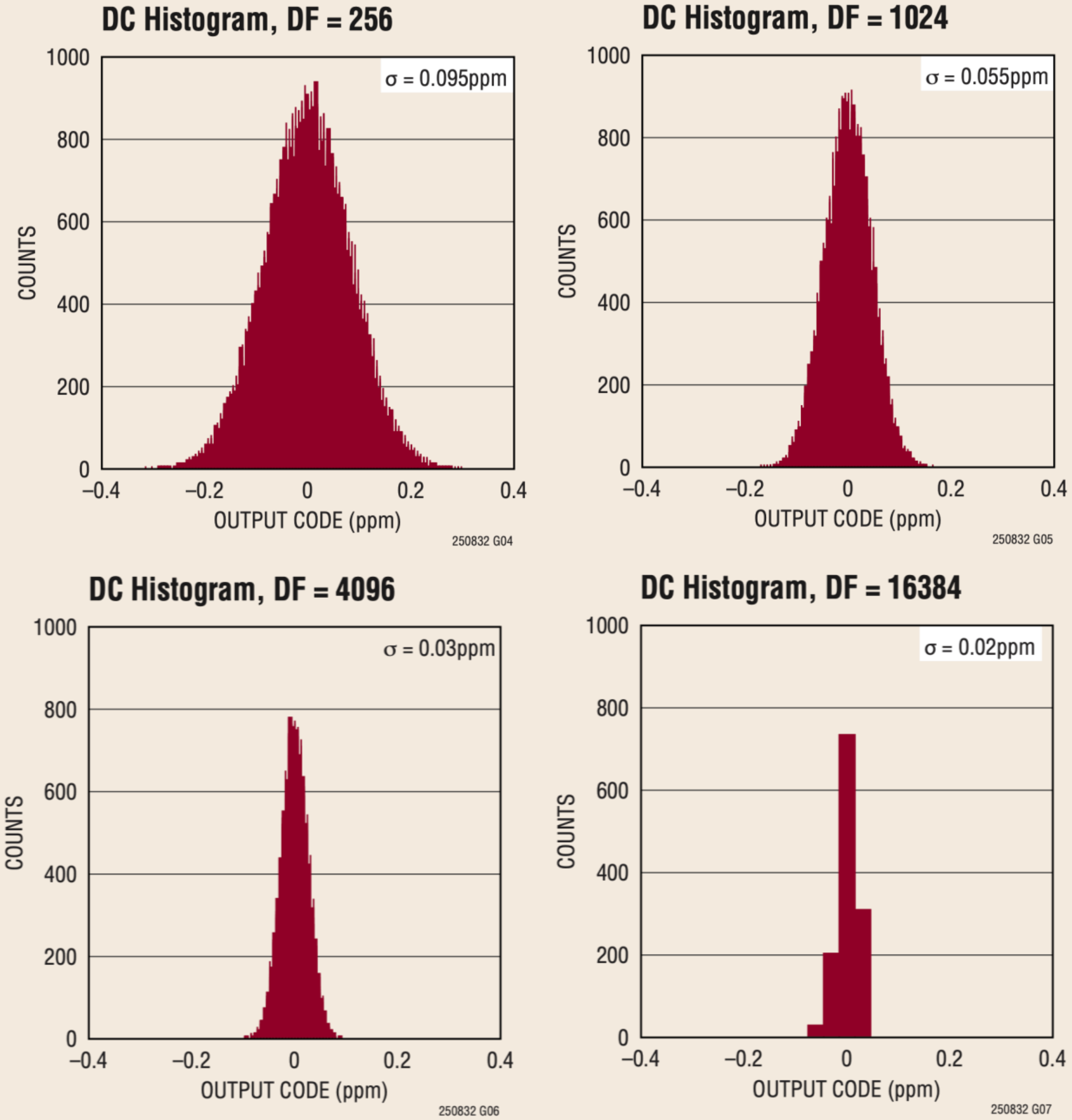
- Convolution with low-pass $h(t)$
- 127 coeff. Blackman-Harris kernel provides 70 dB stop-band attenuation



S. S. Smith, *Digital Signal Processing*, California Technical Publishing, 1997

Down Sampling (Example)

Figures: LTC2508-32 data sheet p.7,
©2016 Linear Technology / Analog Devices


DC Histogram, DF = 256

DC Histogram, DF = 1024

DC Histogram, DF = 4096

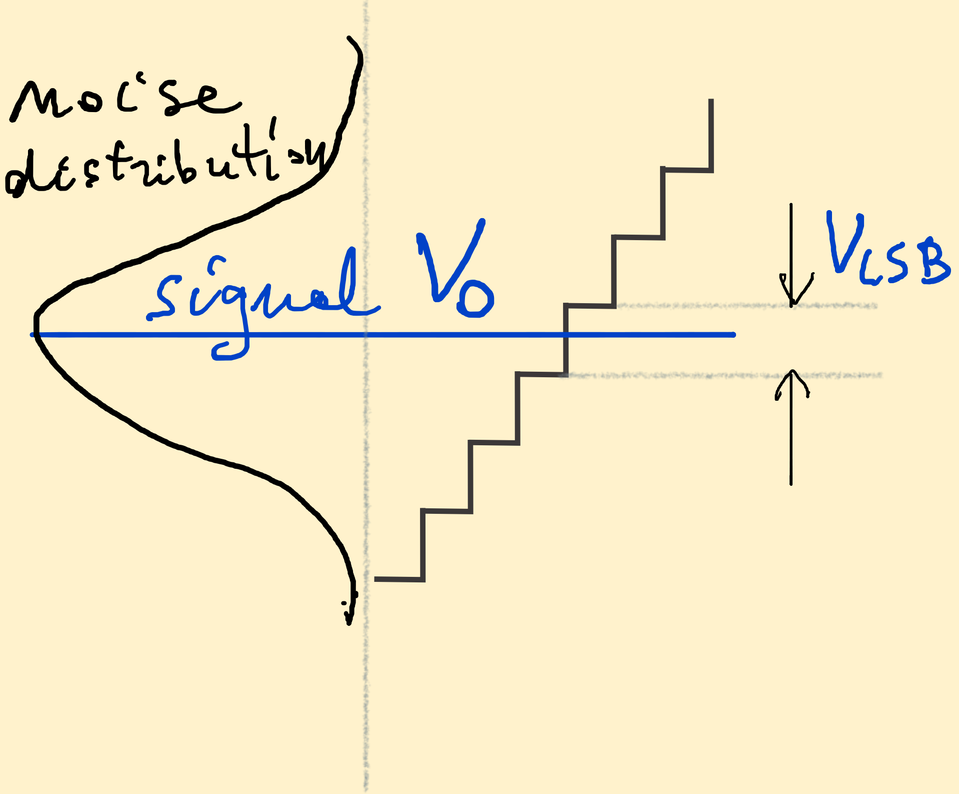
DC Histogram, DF = 16384

- DF is the Decimation Factor
  $\mathrm{DF} = B_{\max}/B\mathrm{Bandwidthratio}$
- A factor 4 in $B_{\max}/B$ results in 1 bit resolution increase

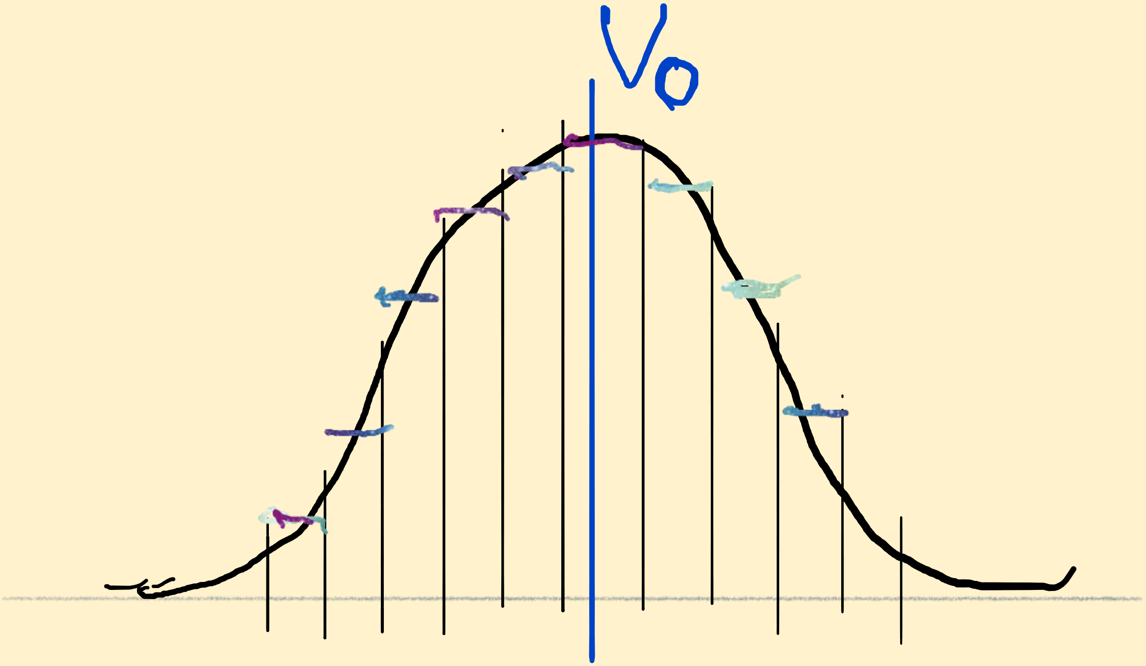
ADS1262, Texas Instrument
LTC2508-32, Linear Technology / Analog Devices

Dithering

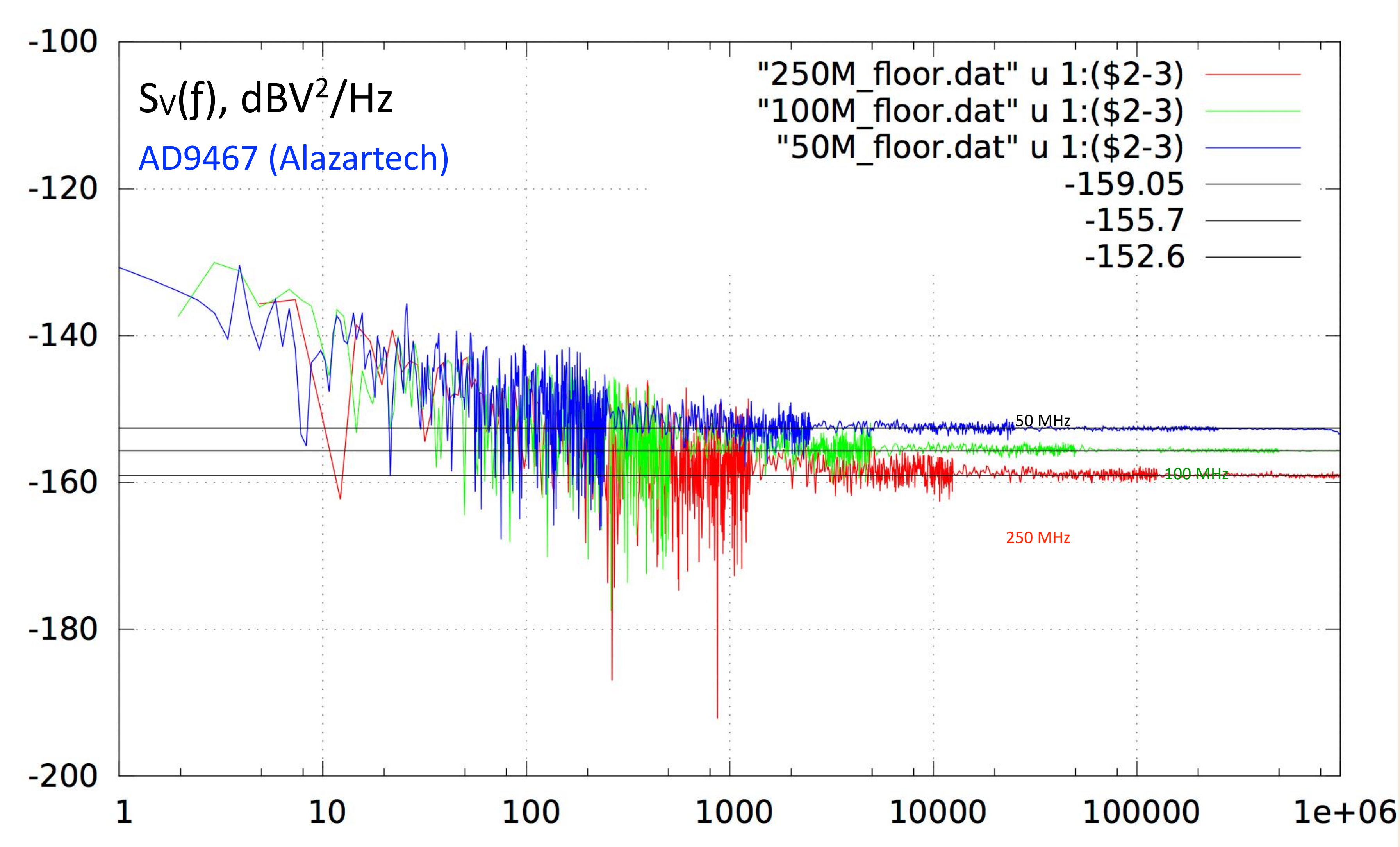
Historical challenge: resolution of a fraction of V_{LSB}



- Add white noise and average
- Estimate the center of the distribution



Sampling Frequency



The observed floor fits the theory

We always use the highest sampling frequency

Selected High-Speed ADCs

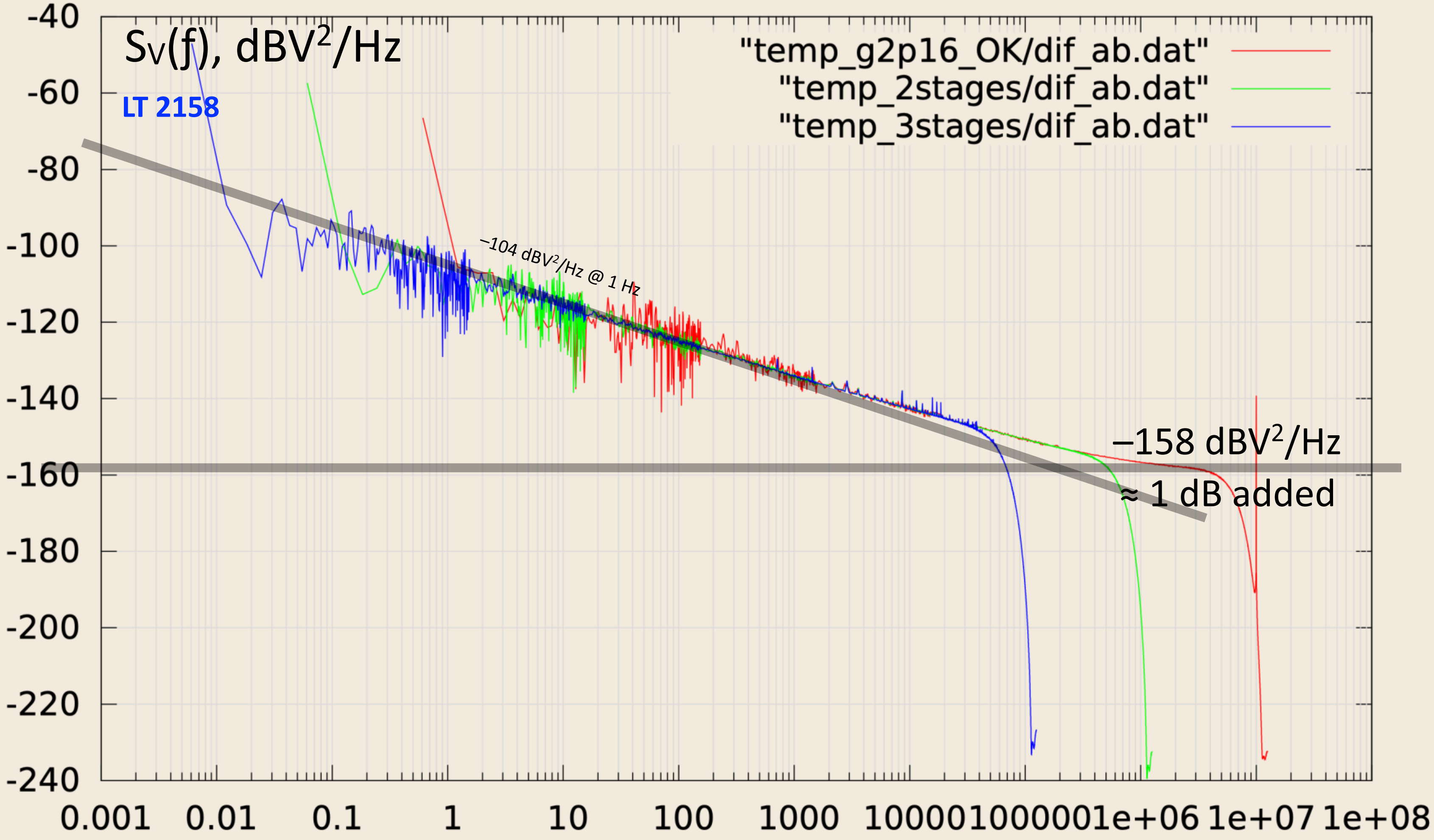
ADC type	AD9467 / Single (Alazartech board)	LTC2145 / Dual Red Pitaya board	LTC2158 / Dual Eval board
Platform	Computer	Zynq (onboard)	Zynq (separated)
Sampling ƒ Input BW	250 MHz 900 MHz	125 MHz 750 MHz	310 MHz 1250 MHz
Bits / ENoB	16 / 12	14 / 12	14 / 12
Expected noise (2 V_{fsr})	–158 dBV^2/Hz	–155 dBV^2/Hz	–159 dBV^2/Hz
Delay & Jitter	1.2 ns & 60 fs	0? & 100 fs diff 0? & 80 fs single	1 ns & 150 fs
Power supply	1.8 V & 3.3 V 1.33 W	1.8 V 190 mW	1.8 V 725 mW

Dissipation is relevant to thermal stability

For reference, 100 fs jitter is equivalent to

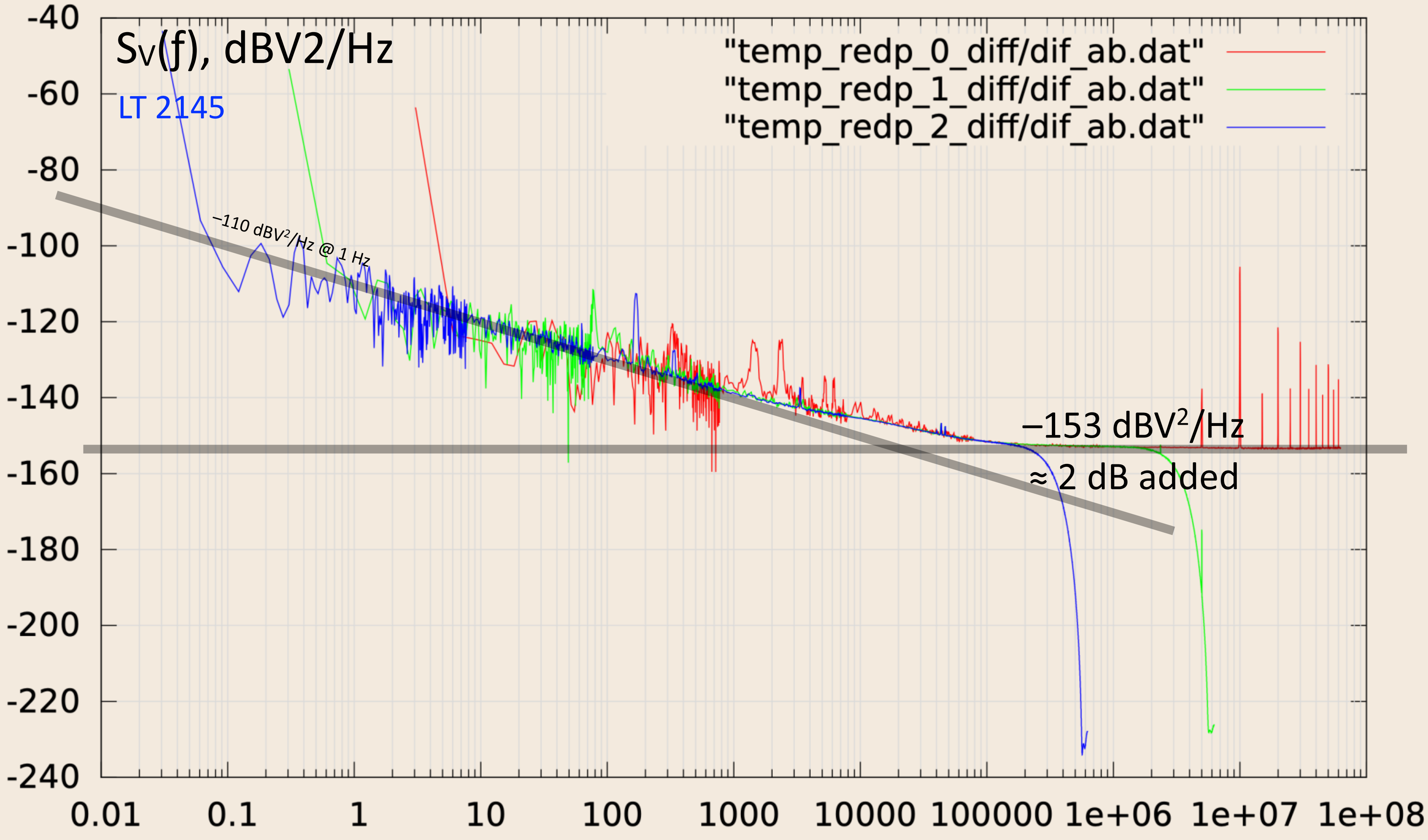
carrier ƒ	φ rms	S$_\varphi$(ƒ) = b$_0$	10 Log$_{10}$[L(ƒ)]
10 MHz	6.3 µrad	4x10^{-18} rad^2/Hz	–177 dBc/Hz
100 MHz	63 µrad	4x10^{-17} rad^2/Hz	–167 dBc/Hz

LT 2158 Noise



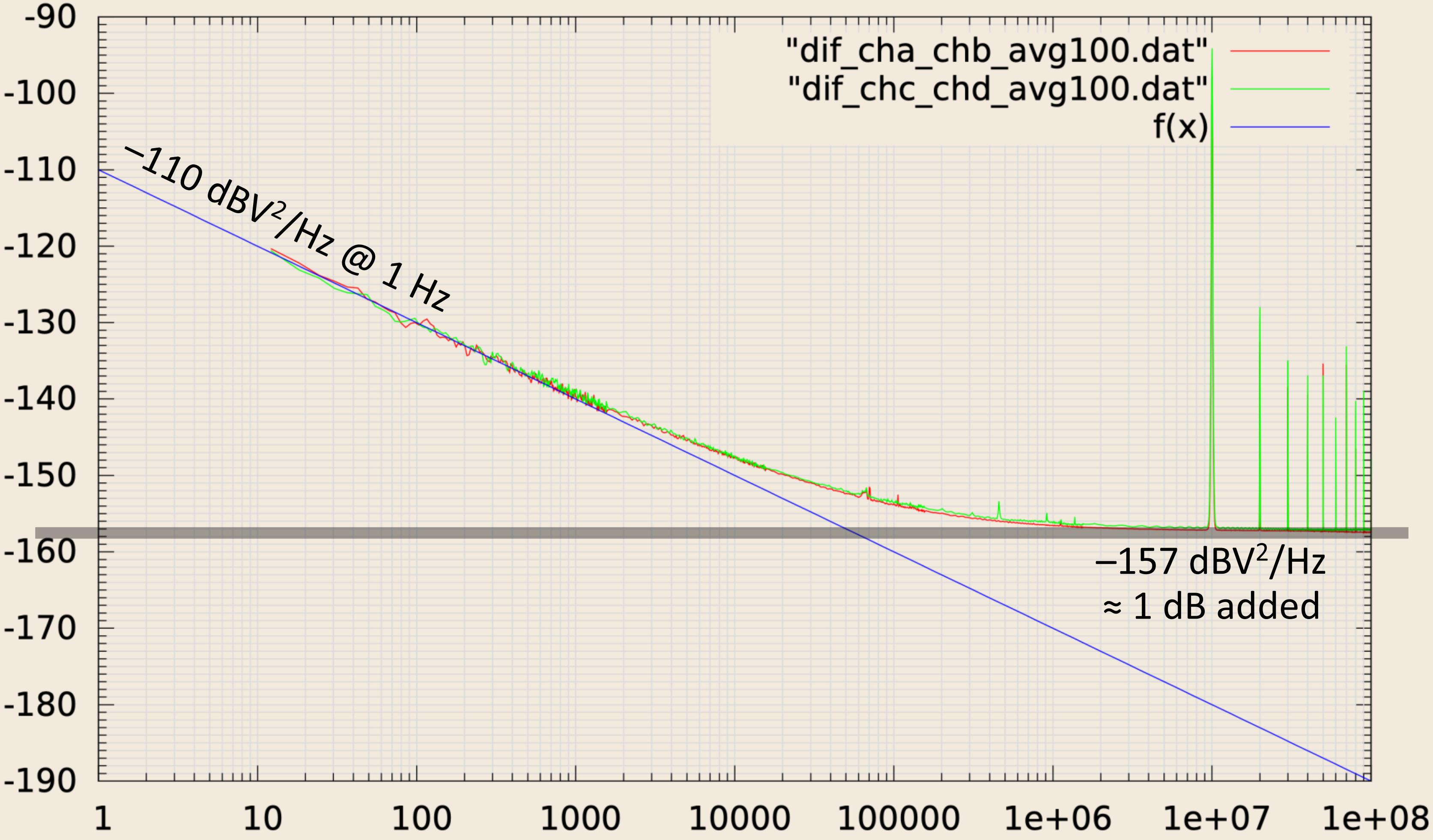
10 MHz, $V_{pp} \approx 0.95\ V_{FSR}$

LT2145 (Red Pitaya) Noise



10 MHz, $V_{pp} \approx 0.95\ V_{FSR}$

AD9467 (Alazartech) Noise



10 MHz, $V_{pp} \approx 0.95\ V_{FSR}$

Skip

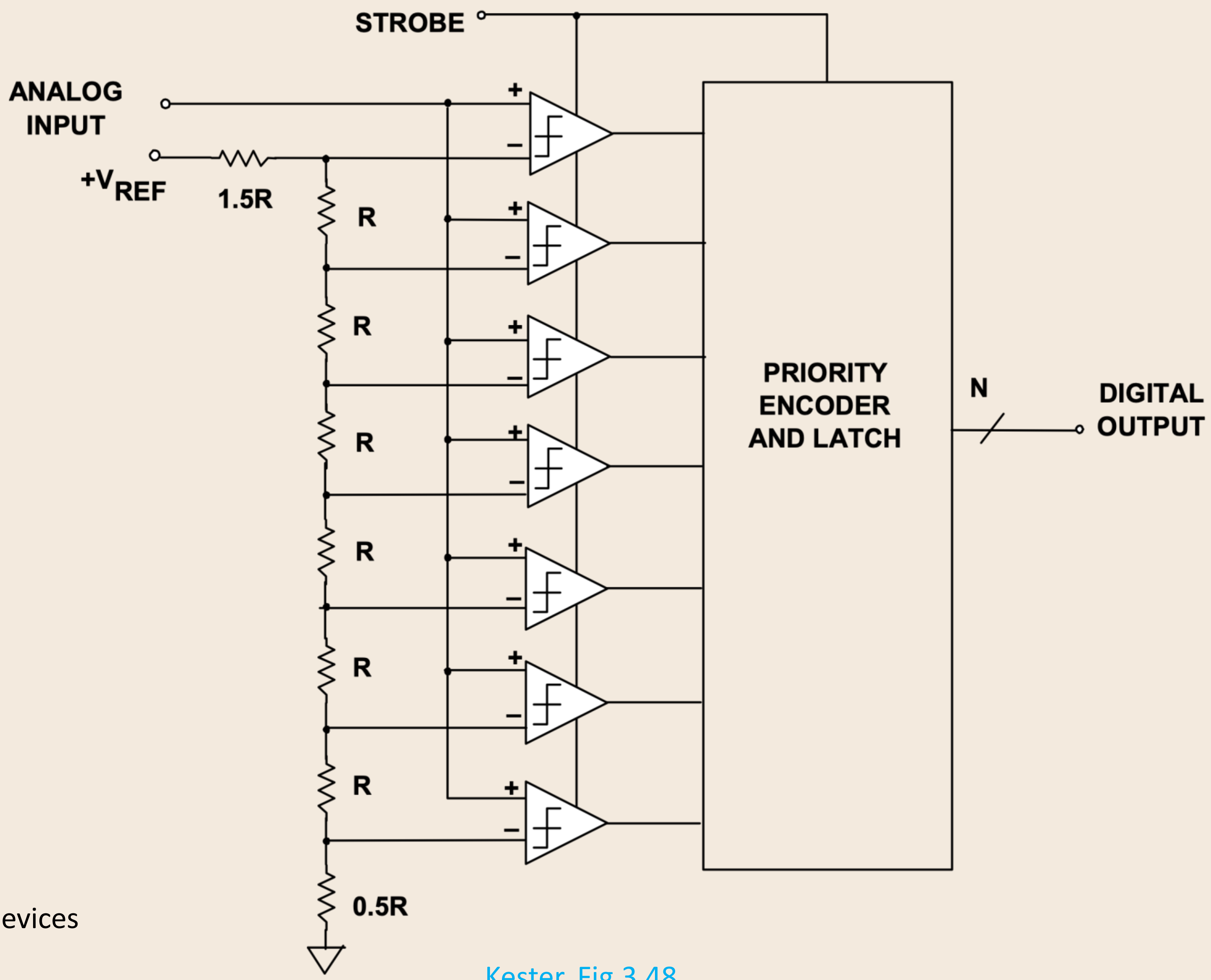
ADC Architectures

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Read it again, again and again

Flash

- Fastest, sub-nanosecond

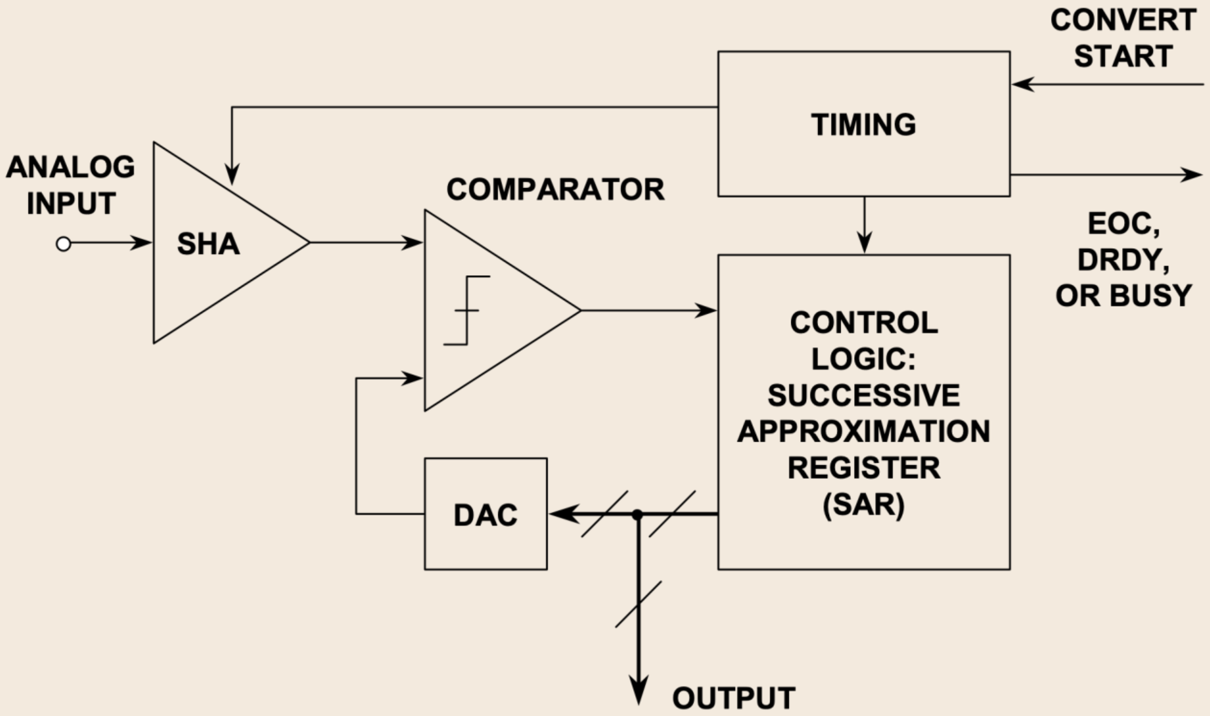


Kester, Fig.3.48

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Successive Approximation (SAR)

- High accuracy
- High resolution, up to 32 bits
- Testing n bits takes n clock cycles
- Latency and downsampling
  - Slow, full accuracy and resolution
  - Moderate, at cost of accuracy
- The internal DAC uses switched capacitors (resistor network was obsoleted long ago)
- Tracking operation possible
  - Faster, but limited slew rate

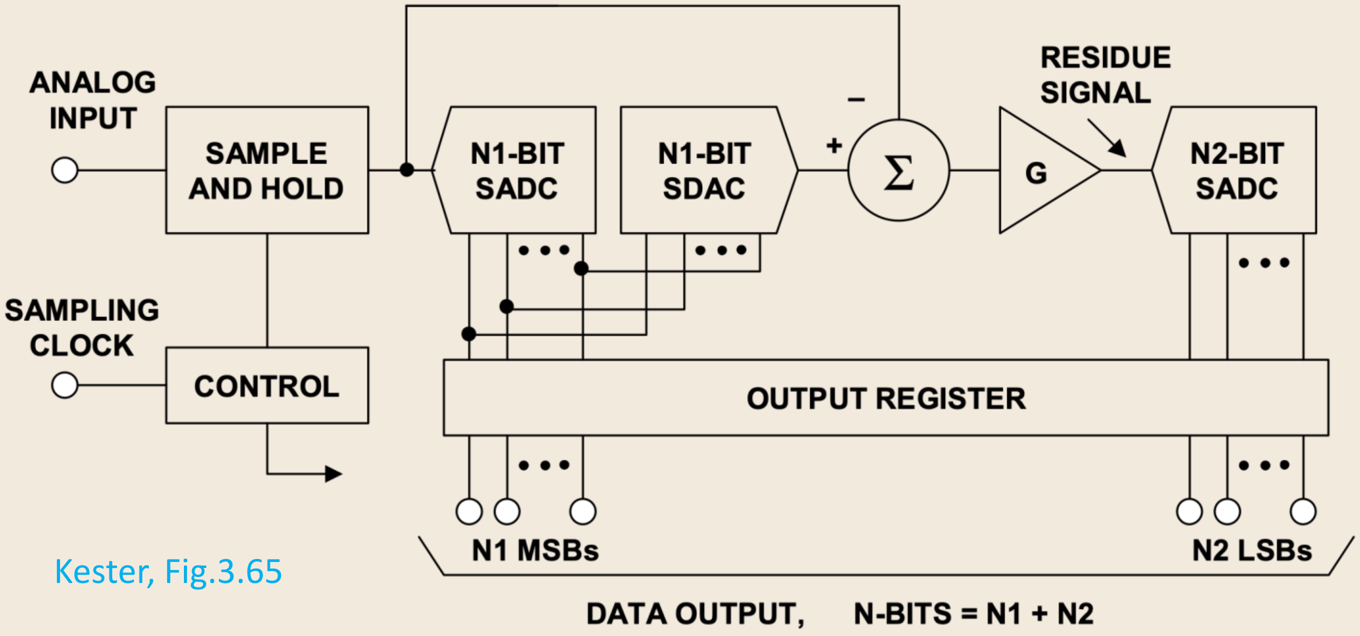


Kester, Fig.3.56

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Subranging

- Pipeline
- Great speed/resolution tradeoff



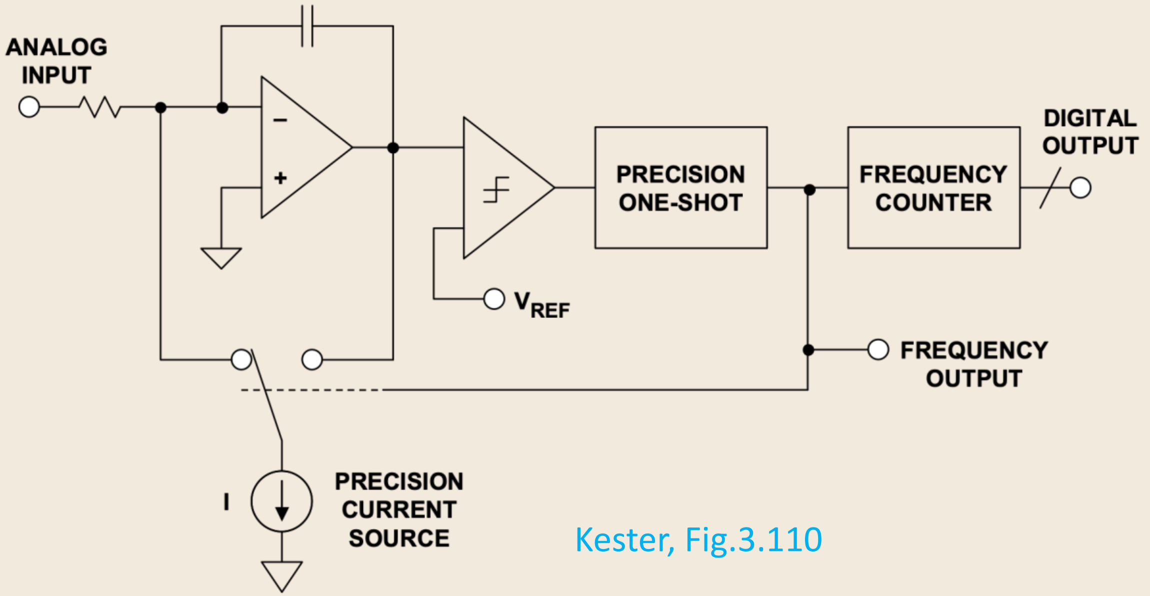
Kester, Fig.3.65

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Counting

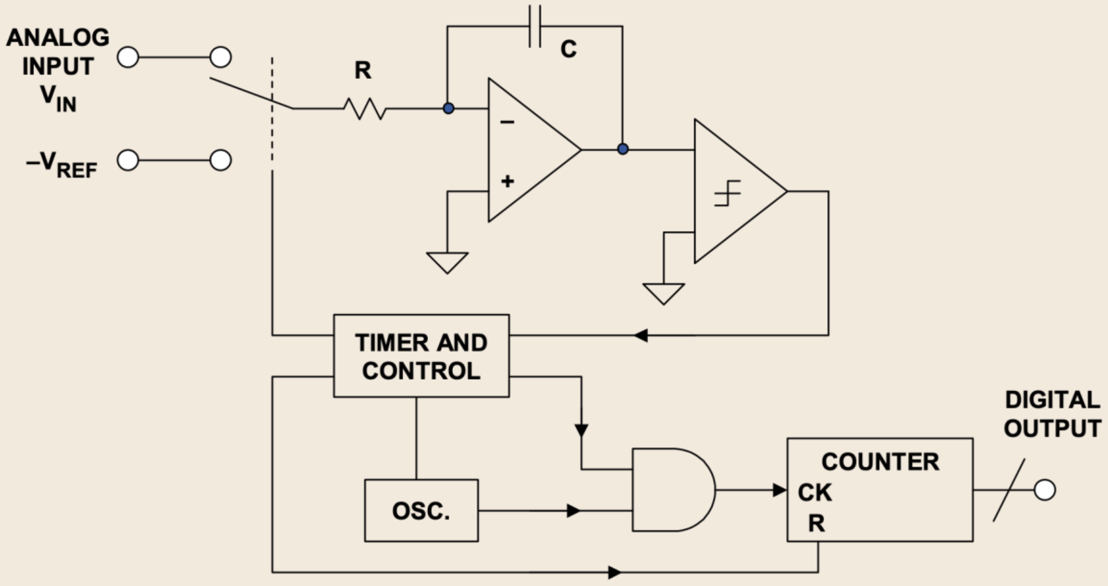
A few techniques – Analog integrator

Voltage-to-frequency converter



Kester, Fig.3.110

Dual slope



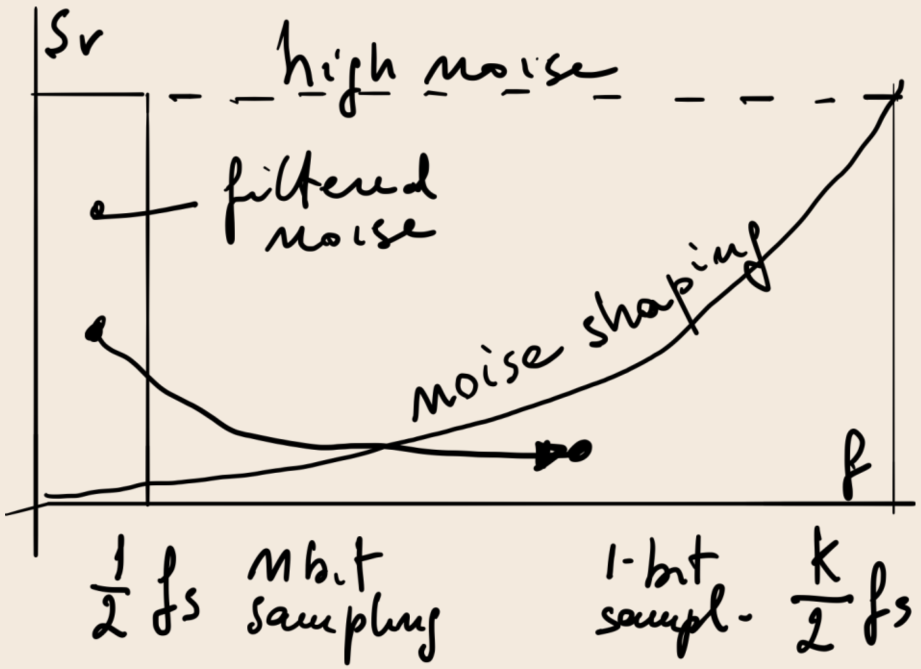
Kester, Fig.3.113

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

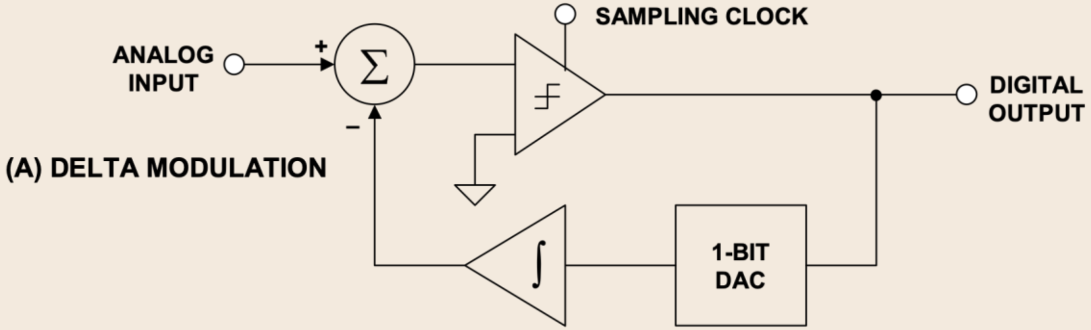
An education version of these converters is in E. Rubiola, *Laboratorio di misure elettroniche* (in Italian), CLUT, Torino, 1993. ISBN 88-7992-081-2

Sigma Delta

- High resolution and low power for cheap
- Simple ideas, but complex mathematics
- Noise shaping



Delta modulation – Kester, Fig.3.119(A)



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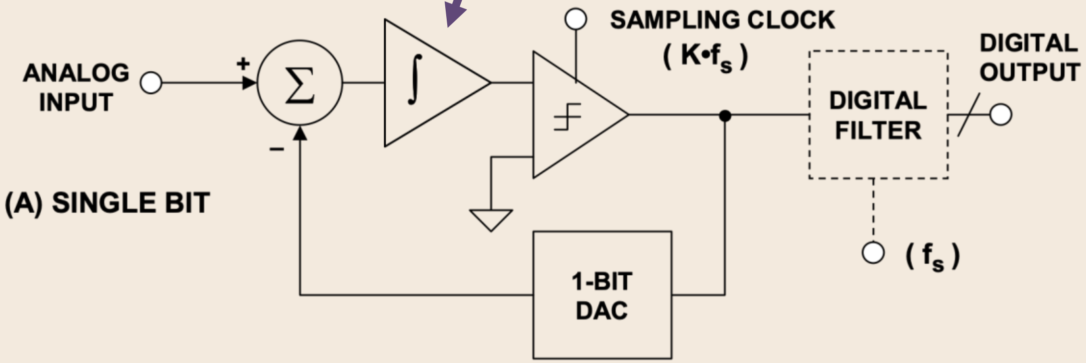

Delta modulation – Kester, Fig.3.121(A)

Featured reading: W Kester (ed), *Analog-Digital Conversion*, Analog Devices 2004, ISBN 0-916550-27-3. ©AD, but free of charge pdf

Lecture 4

Scientific Instruments & Oscillators

Lectures for PhD Students and Young Scientists

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Contents

- Fourier statistics
- The cross spectrum method (theory)
- Applications of the cross spectrum method

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home page http://rubiola.org

Power Spectral Density (PSD) and its Estimation

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

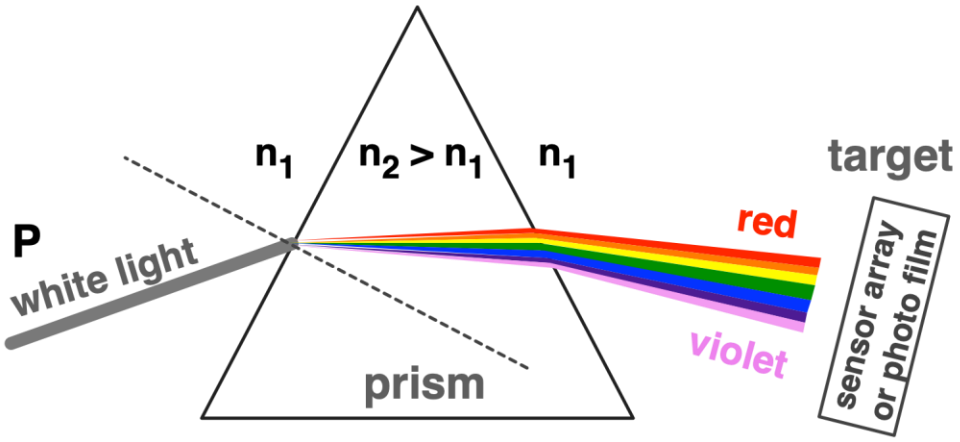
INRiM, Torino, Italy

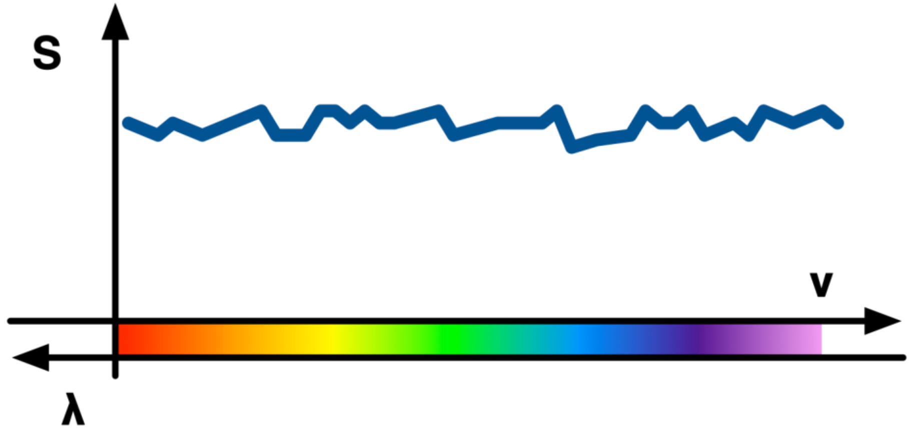
Featured reading: E. Rubiola, F. Vernotte
The cross-spectrum experimental method
https://arxiv.org/abs/1003.0113

home page http://rubiola.org

Physical concept of spectrum

More precisely, Power Spectral Density





- The PSD is the distribution of power vs. frequency (power in 1-Hz bandwidth)
- The PS is the distribution of energy vs. frequency (energy in 1-Hz bandwidth)
- Power (energy) in physics is a square (integrated) quantity
- PSD –> W/Hz, or V^2/Hz, A^2/Hz, rad^2/Hz etc.

$$S_v(f) = \frac{\langle v_B^2(f)\rangle}{B}$$

Discrete: Δf is the resolution
Continuous: $\Delta f \to df$

$$\frac{\text{average power in the bandwidth } B \text{ centered at } f}{\text{bandwidth } B}$$

The power spectral density

for stationary random process $x(t) \leftrightarrow X(f)$

$$\mathcal{C}_x(\tau) = \mathbb{E}\{[\mathrm{x}(t) - \mu][\mathrm{x}(t-\tau) - \mu]^*\}$$

Autocovariance function
Improperly referred to as the correlation, denoted with $R_{xx}(\tau)$

$$\mu = \mathbb{E}\{\mathrm{x}\}$$

$$S(\omega) = \mathcal{F}\{\mathcal{C}(\tau)\} = \int_{-\infty}^{\infty} \mathcal{C}(\tau)\, e^{-i\omega\tau}\ d\tau$$

PSD (two-sided)

Fourier transform of the autocovariance

$$\mathcal{C}_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [x(t) - \mu]\,[x(t-\tau) - \mu]^*\, dt$$

For ergodic process, interchange ensemble and time average
process $\mathrm{x}(t) \rightarrow$ realization $x(t)$

$$S_x^{II}(\omega) = \lim_{T\to\infty} \frac{1}{T} X_T(\omega) X_T^*(\omega) \ = \lim_{T\to\infty} \frac{1}{T} |X_T(\omega)|^2$$

Wiener Khinchin theorem, if the process is stationary and ergodic, $S_x(f)$ can be calculated from the Fourier transform of a realization

In experiments, we use the single-sided PSD averaged on m realizations

$$S^I(f) = 2S^{II}(\omega/2\pi)$$
$$f > 0$$

$$S_x(f) = \frac{2}{T} \langle X_T(f) X_T^*(f) \rangle_m$$

DFT, FFT, FFTW, SFFT

The Discrete Fourier Transform (DFT) approximates the (continuous) FT

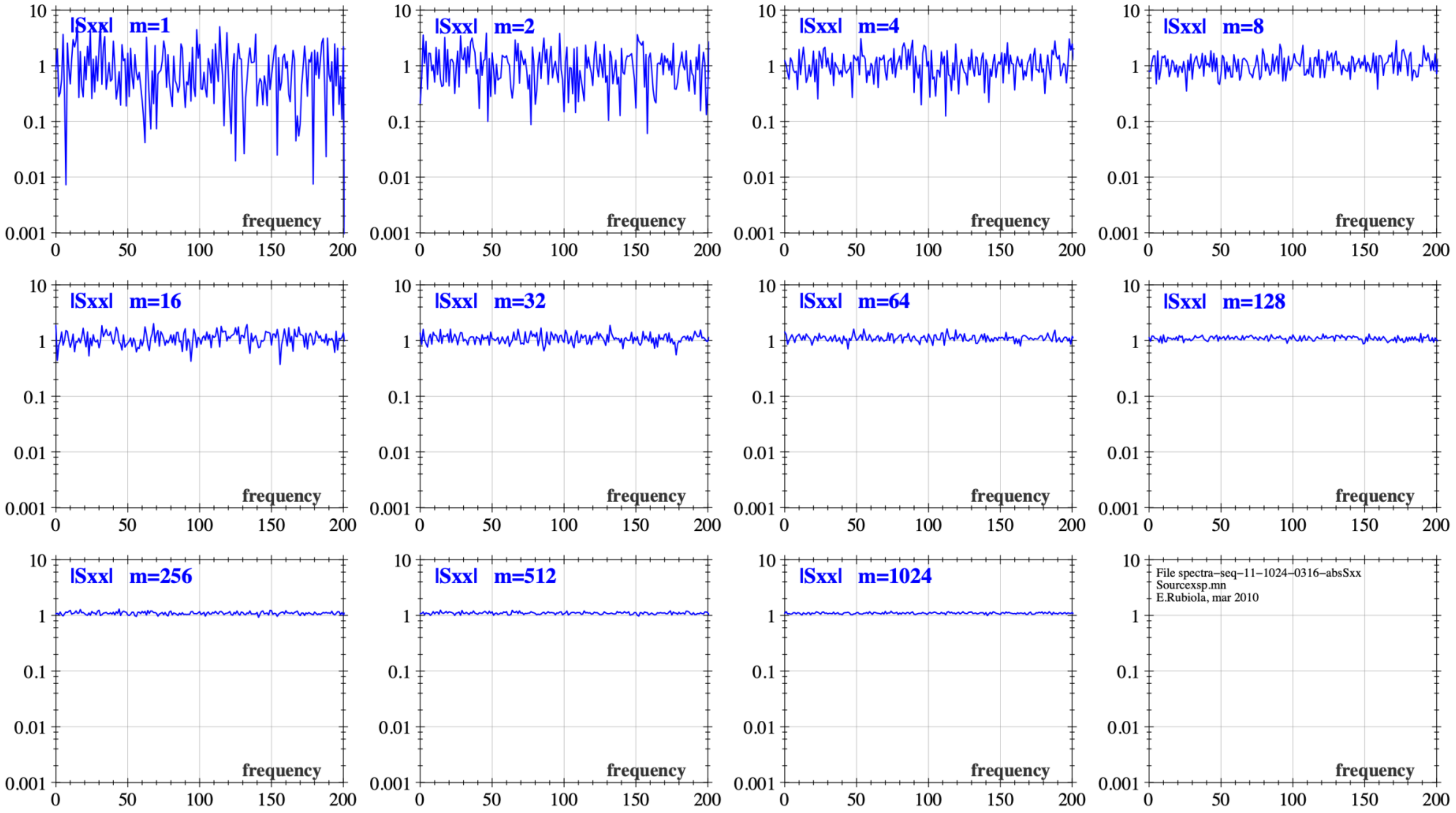
$$X\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} x(kT)e^{i2\pi nk/N}$$

$T =$ sampling interval, $f_s = 1/T$

$N = 0 \ldots N - 1$ integer frequency, $f = n/NT$

- The direct computation of the DFT takes ≈ N2 multiplications
- The FFT is an algorithm for Fast computation of the DFT that takes ≈ N log(N) multiplications
- The FFTW, “the Fastest Fourier Transform in the West,” is an even faster.
  N log(N) multiplications (M. Frigo, S.G. Johnson, MIT)
  See http://fftw.org/
- SFFT “faster-than-fast” Sparse (FFT, D.Katabi, P.Indyk, MIT)
  See http://groups.csail.mit.edu/netmit/sFFT/
- For the general user (does not implement FT algorithms),
  the difference between DFT, FFT, and FFTW is (at most)
  computing time

Estimation of $|S_{xx}(f)|$



Running the measurement, m increases and $S_{xx}(f)$ shrinks => better confidence level

Power spectral density $S_{xx}(f)$

$x(t) \leftrightarrow X(f)$ is white Gaussian noise

Take one frequency, $S(f) \to S$

Same applies to all frequencies

Normalization: in 1 Hz bandwidth

$\mathbb{V}\{X\} = 1$, equally split between $\Re\{\}$ and $\Im\{\}$

thus $\mathbb{V}\{X'\} = \mathbb{V}\{X''\} = 1/2$

$$\begin{aligned}\langle S_{xx}\rangle_m &= \frac{2}{T}\,\langle XX^*\rangle_m \\ &= \frac{2}{T}\,\langle (X' + iX'') \times (X' - iX'')\rangle_m \\ &= \frac{2}{T}\,\langle (X')^2 + (X'')^2\rangle_m\end{aligned}$$

white, Gaussian,
$\mu = 0, \;\; \sigma^2 = 1/2$

white, χ^2 , 2 DF
$\mu = 1, \;\; \sigma^2 = 1$

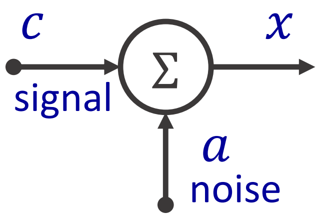
white, χ^2 , $2m$ DF
$\mu = 1, \;\; \sigma^2 = 1/m$

Conclusion

$$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{1}{m}}$$

the $S_{xx}(f)$ track shrinks as $1/\sqrt{m}$

PSD $S_{xx}(f)$

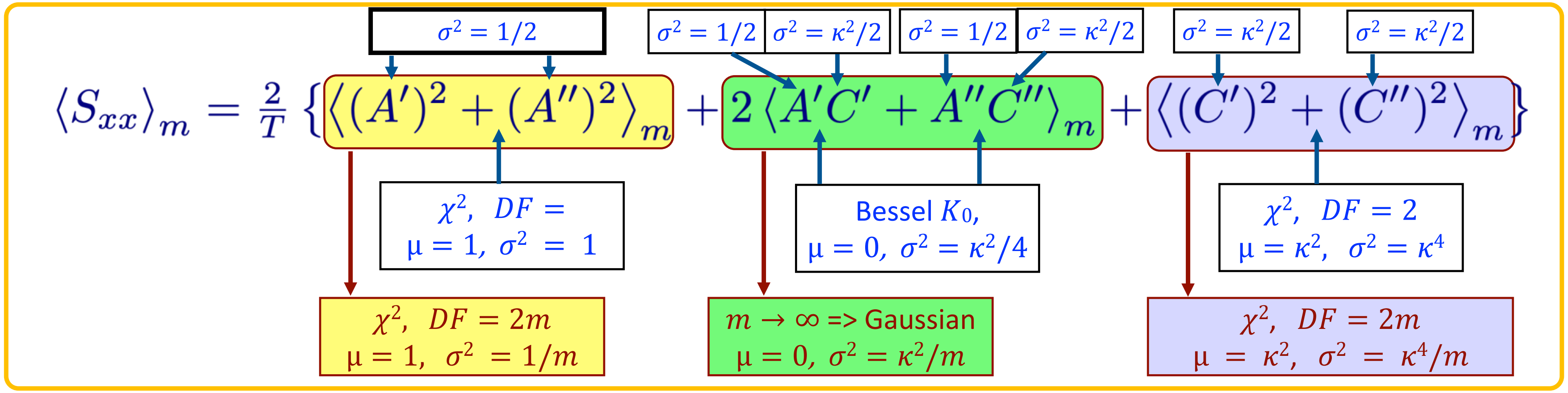


Normalization: in 1 Hz bandwidth $\mathbb{V}\{A\}=1,\ \mathbb{V}\{C\}=\kappa^2$
$\mathbb{V}\{A'\}=\mathbb{V}\{A''\}=1/2$ and $\mathbb{V}\{C'\}=\mathbb{V}\{C''\}=\kappa^2/2$

$$\langle S_{xx}\rangle_m = \frac{2}{T}\langle XX^*\rangle_m = \frac{2}{T}\langle (X'+iX'')\times(X'-iX'')\rangle_m$$
$$X=(C'+iC'')+(A'+iA'')$$

$\Re\{|S_{xx}|\}\rightarrow$

$\Im\{|S_{xx}|\}=0$



$$\langle S_{xx}\rangle_m = \frac{2}{T}\left\{\langle (A')^2+(A'')^2\rangle_m + 2\,\langle A'C'+A''C''\rangle_m + \langle (C')^2+(C'')^2\rangle_m\right\}$$

$\sigma^2=1/2$

χ^2, $DF =$
$\mu=1,\ \sigma^2\ =\ 1$

χ^2, $DF=2m$
$\mu=1,\ \sigma^2\ =1/m$

$\sigma^2=1/2$ | $\sigma^2=\kappa^2/2$ | $\sigma^2=1/2$ | $\sigma^2=\kappa^2/2$

Bessel K_0,
$\mu=0,\ \sigma^2=\kappa^2/4$

$m\to\infty$ => Gaussian
$\mu=0,\ \sigma^2=\kappa^2/m$

$\sigma^2=\kappa^2/2$ | $\sigma^2=\kappa^2/2$

χ^2, $DF=2$
$\mu=\kappa^2,\ \ \sigma^2=\kappa^4$

χ^2, $DF=2m$
$\mu\ =\ \kappa^2,\ \sigma^2\ =\ \kappa^4/m$

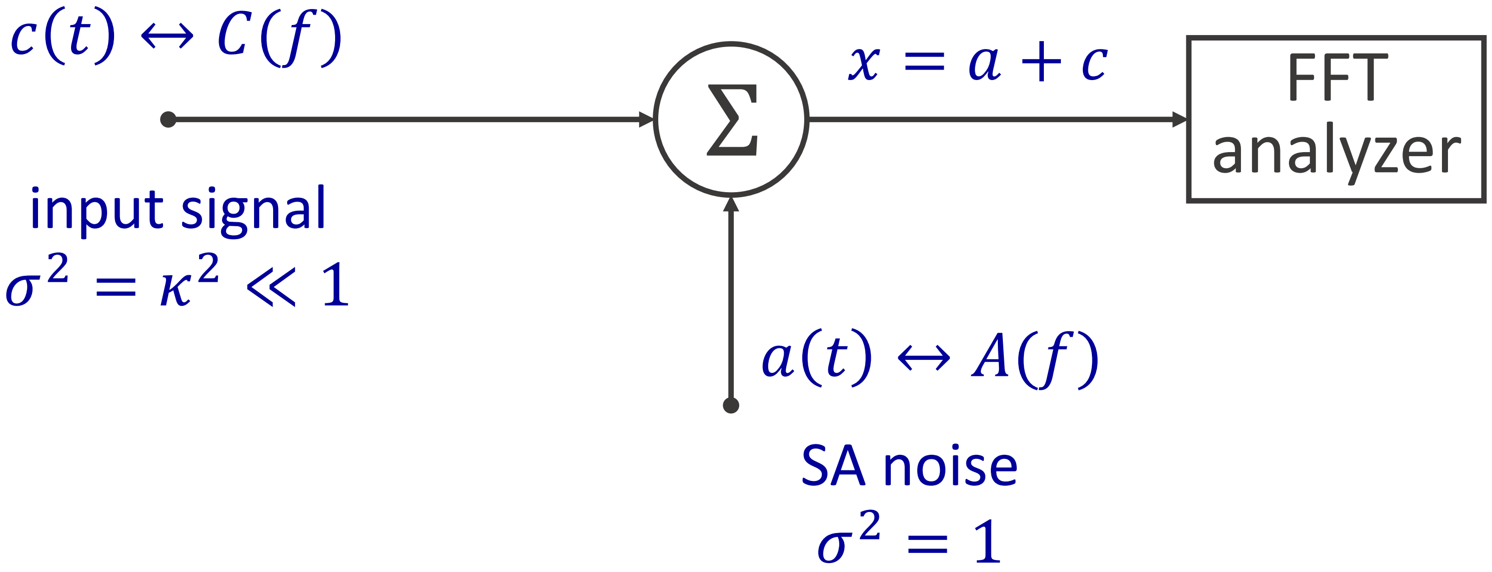
$$\mu = 1+\kappa^2 \qquad \sigma^2=\frac{1+\kappa^2+\kappa^4}{m} \qquad \frac{\sigma}{\mu}=\sqrt{\frac{1+\kappa^2+\kappa^4}{m}}\,\frac{1}{1+\kappa^2}$$

$$\frac{\sigma}{\mu}\simeq\frac{1}{\sqrt{m}}\left[1-\frac{\kappa^2}{2}\right],\qquad \kappa\ll 1$$

the track
shrinks as $1/\sqrt{m}$

$$\frac{\sigma}{\mu}\simeq\frac{1}{\sqrt{m}}\left[1-\frac{1}{2\kappa^2}\right],\qquad \kappa\gg 1$$

Measurement of a small signal



$c(t) \leftrightarrow C(f)$

input signal

$\sigma^2 = \kappa^2 \ll 1$

Σ

$x = a + c$

FFT analyzer

$a(t) \leftrightarrow A(f)$

SA noise

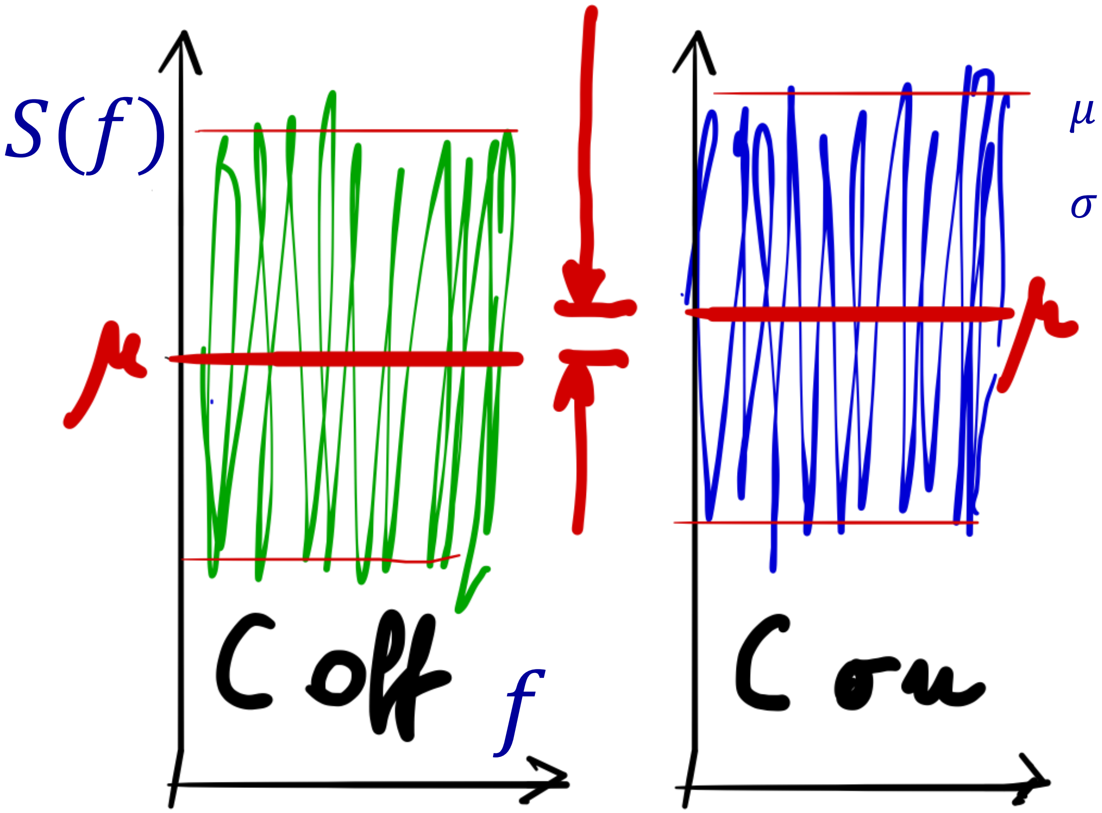
$\sigma^2 = 1$

$C = 0$	$C \neq 0$
$\mu = 1$	$\mu = 1 + \kappa^2$
$\sigma^2 = \dfrac{1}{m}$	$\sigma^2 = \dfrac{1 + \kappa^2 + \kappa^4}{m}$

The Dicke radiometer

Small m

Contrast not detected

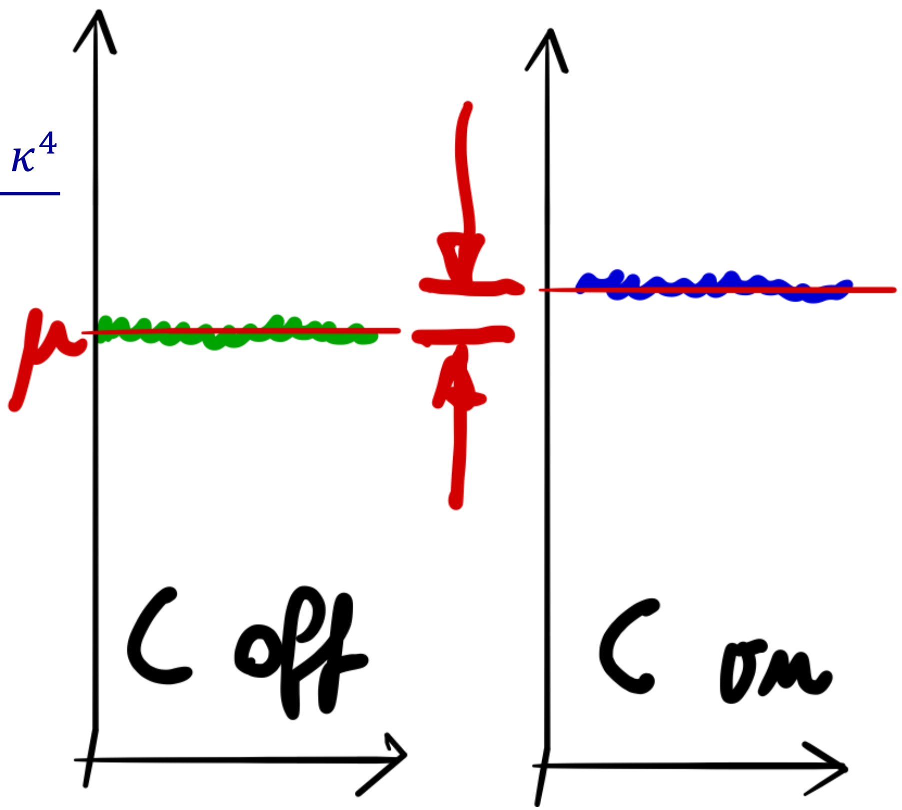


$$\mu = 1 + \kappa^2$$

$$\sigma^2 = \frac{1 + \kappa^2 + \kappa^4}{m}$$

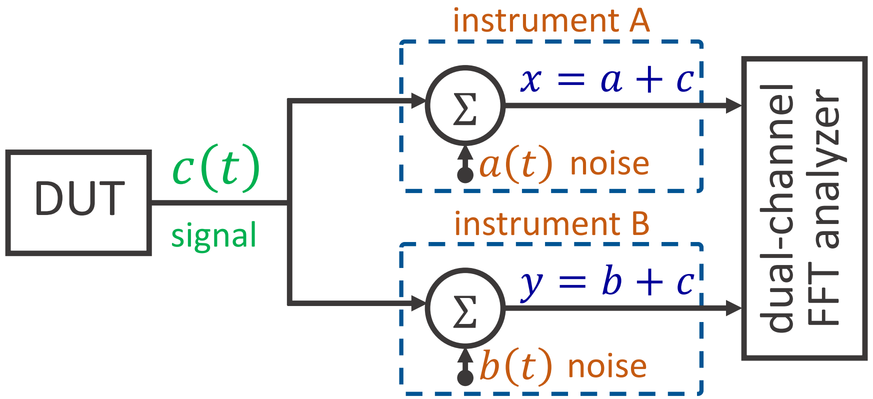
Large m

Contrast well detected

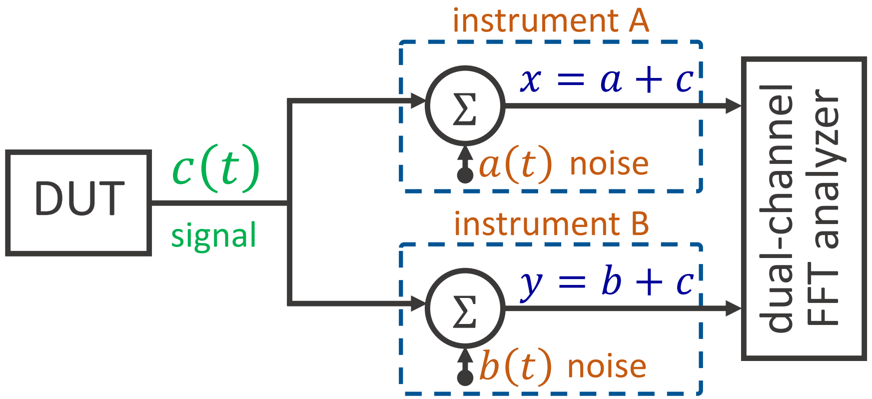


Historical reading: R. H. Dicke, The Measurement of Thermal Radiation at Microwave Frequencies, RSI 17(7) p.268-275, July 1946

Cross Spectrum Theory



Correlation Measurements



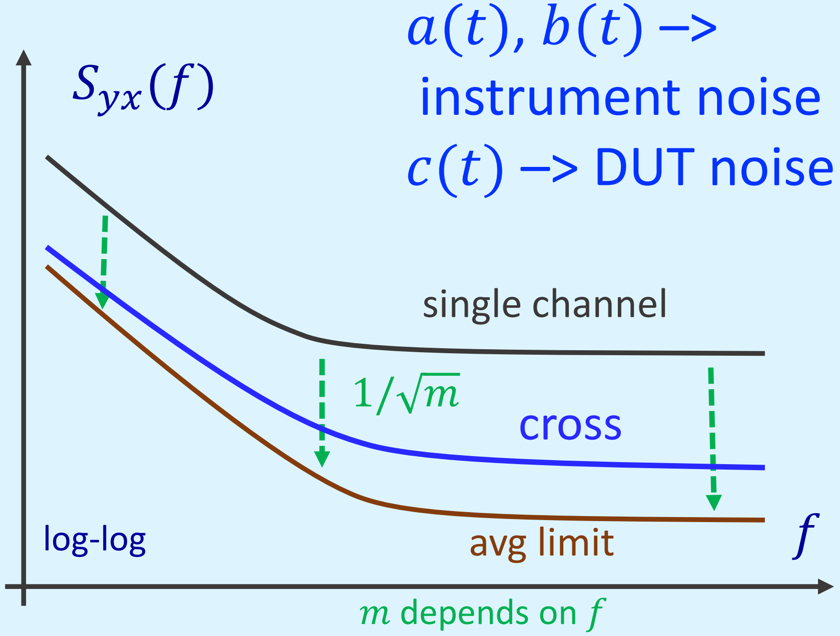
also crosstalk d(t)

Read the tutorial

E. Rubiola, F. Vernotte, The cross-spectrum experimental method, February 2010, arXiv:1003.0113 [physics.ins-det]

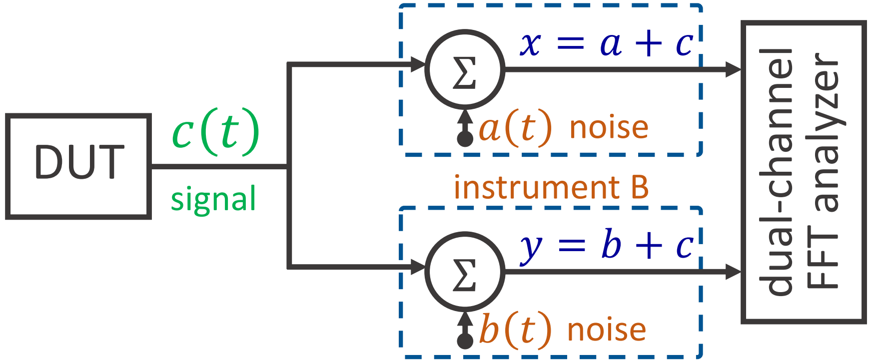
Two separate instruments measure the same DUT.
Only the DUT noise is common

noise measurements		
DUT noise, normal use	a, b, c	instrument noise, DUT noise
background, ideal case	a, b $c = 0$	instrument noise, no DUT
background, real case	a, b $d \neq 0$	c is the correlated instrument noise Zero DUT noise



$a(t), b(t)$ –> instrument noise
$c(t)$ –> DUT noise

Cross PSD $S_{yx}(f)$ – Simplified



$$S_{yx} = \frac{2}{T}\langle (B+C)(A+C)^* \rangle_m$$

$$= \frac{2}{T}\langle \cancel{BA^*} + \cancel{BC^*} + \cancel{CA^*} + CC^* \rangle_m$$

rejected $\propto 1/\sqrt{m}$

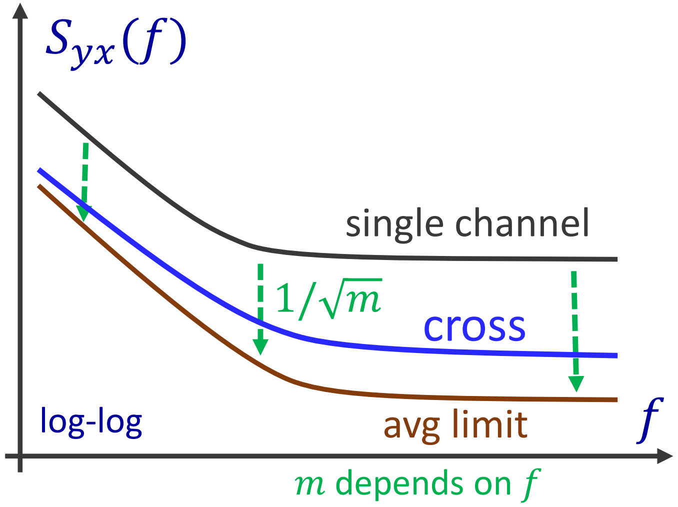
$$\mathbb{E}\{S_{yx}\} = \frac{2}{T}\langle CC^* \rangle_m = S_c \qquad S_c \in \mathbb{R}$$

$$\mathbb{V}\left\{\langle S_{yx}\rangle_m\right\} = \frac{1}{m}$$

The $\widehat{S_{yx}} = |S_{yx}|$ estimator takes in the full noise

$$\mathbb{V}\left\{\langle \Re\{S_{yx}\}\rangle_m\right\} = \frac{1}{2m}$$

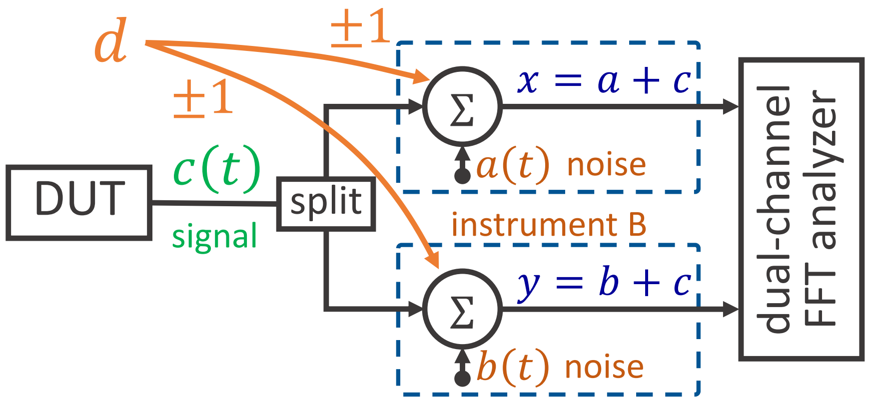
The $\widehat{S_{yx}} = \Re\{S_{yx}\}$ estimator takes in half the noise



Read the tutorial

E. Rubiola, F. Vernotte, The cross-spectrum experimental method, February 2010, arXiv:1003.0113 [physics.ins-det]

A correlated disturbing term



Same role of $c(t)$, but for the sign ς

$$S_{yx} = \frac{2}{T}[B + C + \varsigma_y D][A + C + \varsigma_x D]^*$$

After averaging

$$S_{yx} \to S_c + \overset{\pm 1}{\varsigma} S_d$$

DUT spectrum

bias

Also $\Re\{S_{yx}\} \to S_c + \varsigma S_d$ and $\Im\{S_{yx}\} \to 0$

$\varsigma > 0$ –> noise over-estimation

- We may accept this

$\varsigma < 0$ –> noise under-estimation

- May be embarrassing

$S_c + \varsigma S_d < 0$ –> nonsense

- The disturbing term prevail

The common superstition that

- The instrument adds its own noise
- Over-estimation of the DUT noise

is wrong in the case of cross spectrum (and covariances)

$S_{yx}(f)$ with a correlated term

A, B —> instrument background
C —> DUT noise
channel 1 $X = A + C$
channel 2 $Y = B + C$
A, B, C are independent Gaussian processes
$\Re\{\}$ and $\Im\{\}$ are independent Gaussian processes

Cross-Spectrum

$$\langle S_{yx}\rangle_m = \frac{2}{T}\langle Y X^*\rangle_m = \frac{2}{T}\langle (Y' + iY'') \times (X' - iX'')\rangle_m$$

Expand using

$$X = (A' + iA'') + (C' + iC'') \qquad \text{and} \qquad Y = (B' + iB'') + (C' + iC'')$$

Split S_{yx} into three sets

$$\langle S_{yx}\rangle_m = \big[\,\langle S_{yx}\rangle_m\,\big]_{\text{instr}} + \big[\,\langle S_{yx}\rangle_m\,\big]_{\text{mixed}} + \big[\,\langle S_{yx}\rangle_m\,\big]_{\text{DUT}}$$

background only

background and DUT noise

DUT noise only

Normalization: in 1 Hz bandwidth

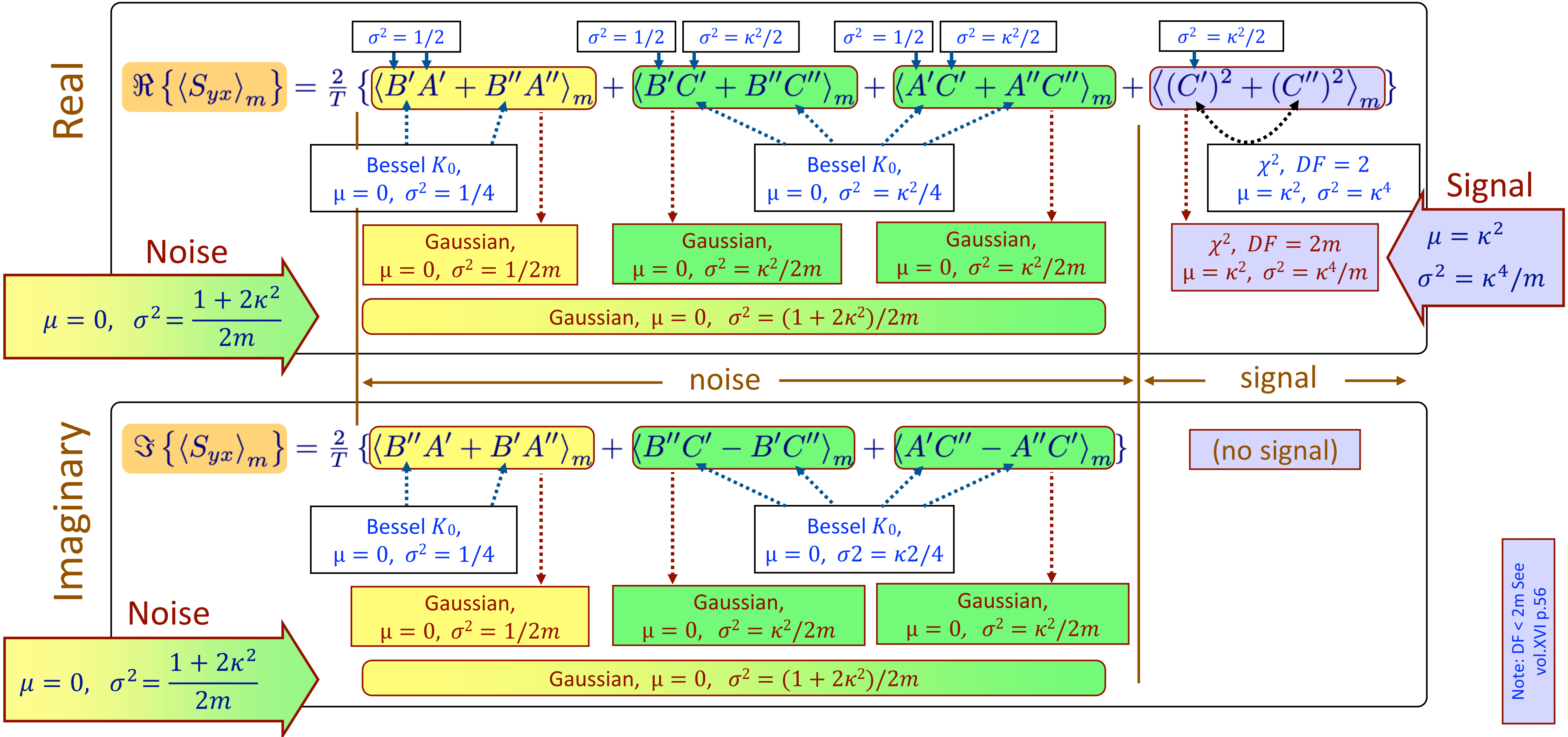
$$\mathbb{V}\{A\} = \mathbb{V}\{B\} = 1$$
$$\mathbb{V}\{A'\} = \mathbb{V}\{A''\} = \mathbb{V}\{B'\} = \mathbb{V}\{B''\} = 1/2$$

$$\mathbb{V}\{C\} = \kappa^2$$
$$\mathbb{V}\{C'\} = \mathbb{V}\{C''\} = \kappa^2/2$$

... and work it out !!!

S_{yx} with correlated term $\kappa \neq 0$

All the DUT signal goes in $\Re\{S_{yx}\}$, while $\Im\{S_{yx}\}$ contains only noise



Real

$\sigma^2 = 1/2$ $\quad$ $\sigma^2 = 1/2$ $\quad$ $\sigma^2 = \kappa^2/2$ $\quad$ $\sigma^2 = 1/2$ $\quad$ $\sigma^2 = \kappa^2/2$ $\quad$ $\sigma^2 = \kappa^2/2$

$$\Re\left\{\langle S_{yx}\rangle_m\right\} = \frac{2}{T}\left\{\langle B'A' + B''A''\rangle_m + \langle B'C' + B''C''\rangle_m + \langle A'C' + A''C''\rangle_m + \langle (C')^2 + (C'')^2\rangle_m\right\}$$

Bessel K_0,
μ $= 0,\ \sigma^2 = 1/4$

Bessel K_0,
μ $= 0,\ \sigma^2 = \kappa^2/4$

$\chi^2,\ DF = 2$
μ $= \kappa^2,\ \sigma^2 = \kappa^4$

Gaussian,
μ $= 0,\ \sigma^2 = 1/2m$

Gaussian,
μ $= 0,\ \sigma^2 = \kappa^2/2m$

Gaussian,
μ $= 0,\ \sigma^2 = \kappa^2/2m$

$\chi^2,\ DF = 2m$
μ $= \kappa^2,\ \sigma^2 = \kappa^4/m$

Gaussian, μ $= 0,\ \sigma^2 = (1 + 2\kappa^2)/2m$

Noise

$$\mu = 0,\quad \sigma^2 = \frac{1 + 2\kappa^2}{2m}$$

Signal

$$\mu = \kappa^2$$
$$\sigma^2 = \kappa^4/m$$

noise

signal

Imaginary

$$\Im\left\{\langle S_{yx}\rangle_m\right\} = \frac{2}{T}\left\{\langle B''A' + B'A''\rangle_m + \langle B''C' - B'C''\rangle_m + \langle A'C'' - A''C'\rangle_m\right\}$$

(no signal)

Bessel K_0,
μ $= 0,\ \sigma^2 = 1/4$

Bessel K_0,
μ $= 0,\ \sigma 2 = \kappa 2/4$

Gaussian,
μ $= 0,\ \sigma^2 = 1/2m$

Gaussian,
μ $= 0,\ \sigma^2 = \kappa^2/2m$

Gaussian,
μ $= 0,\ \sigma^2 = \kappa^2/2m$

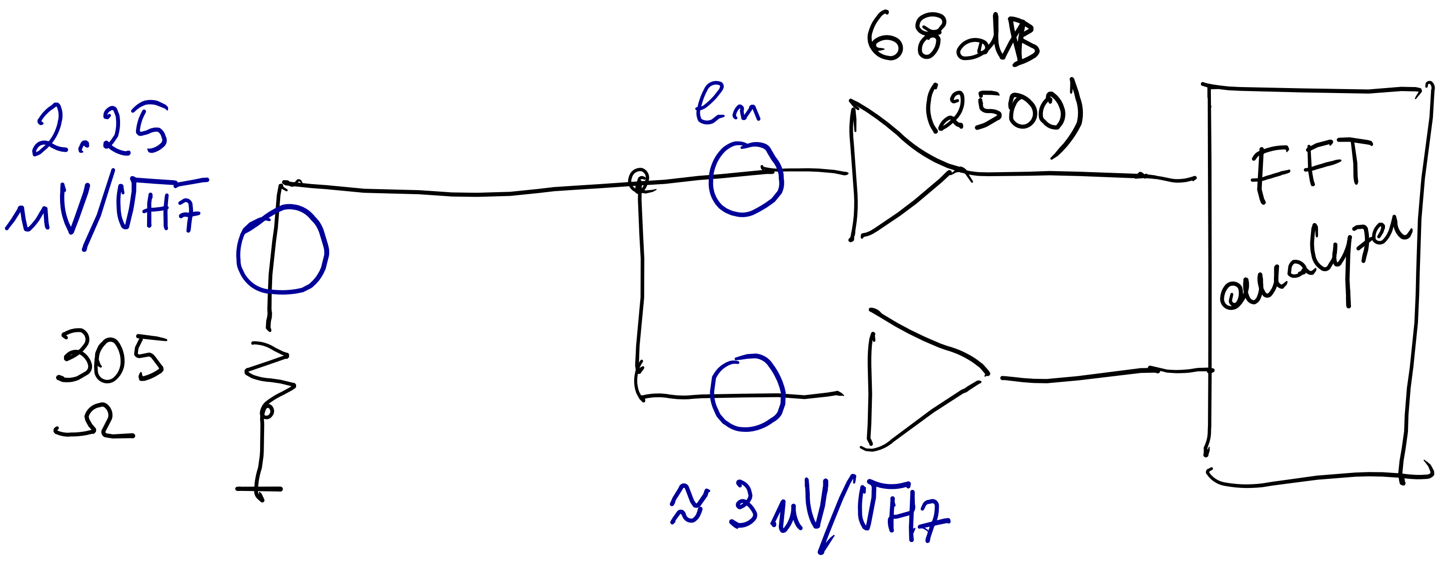
Gaussian, μ $= 0,\ \sigma^2 = (1 + 2\kappa^2)/2m$

Noise

$$\mu = 0,\quad \sigma^2 = \frac{1 + 2\kappa^2}{2m}$$

Note: DF < 2m See
vol.XVI p.56

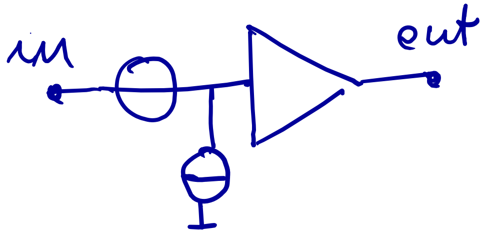
Example / Experiment



Rothe Dahlke model

e_n statistically indep.

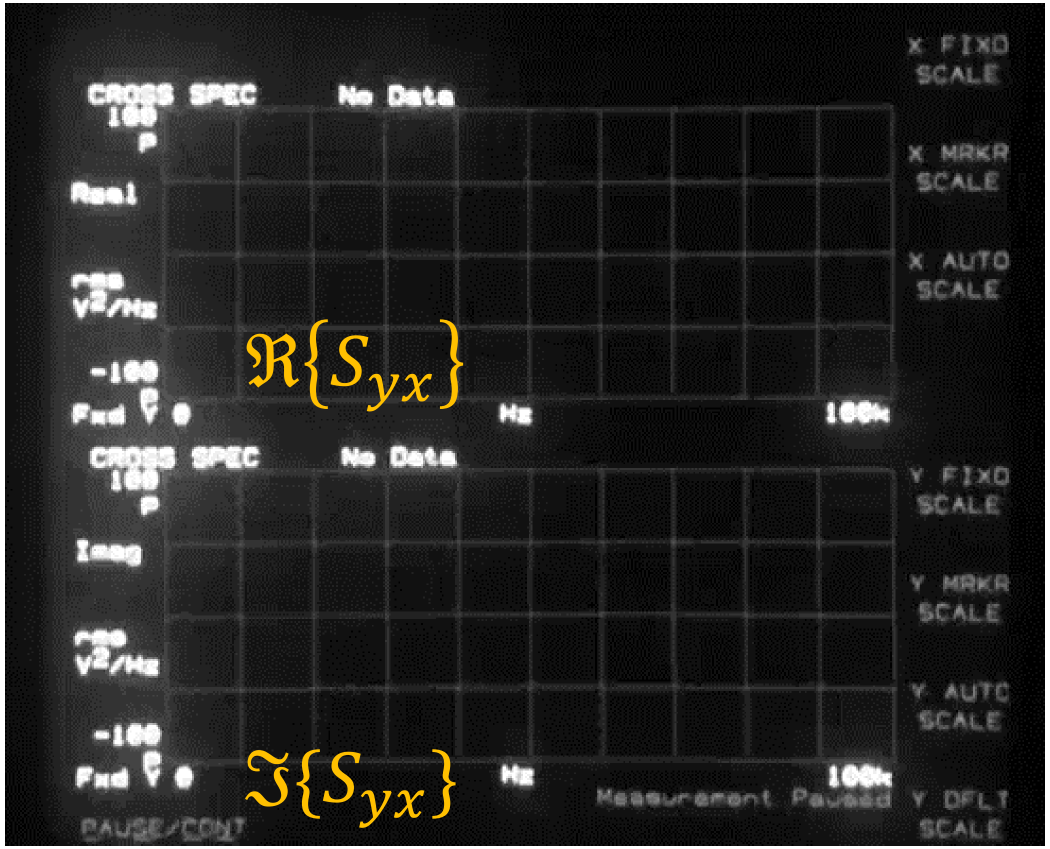
$R\,i_n \to 0$ JFET amplifier



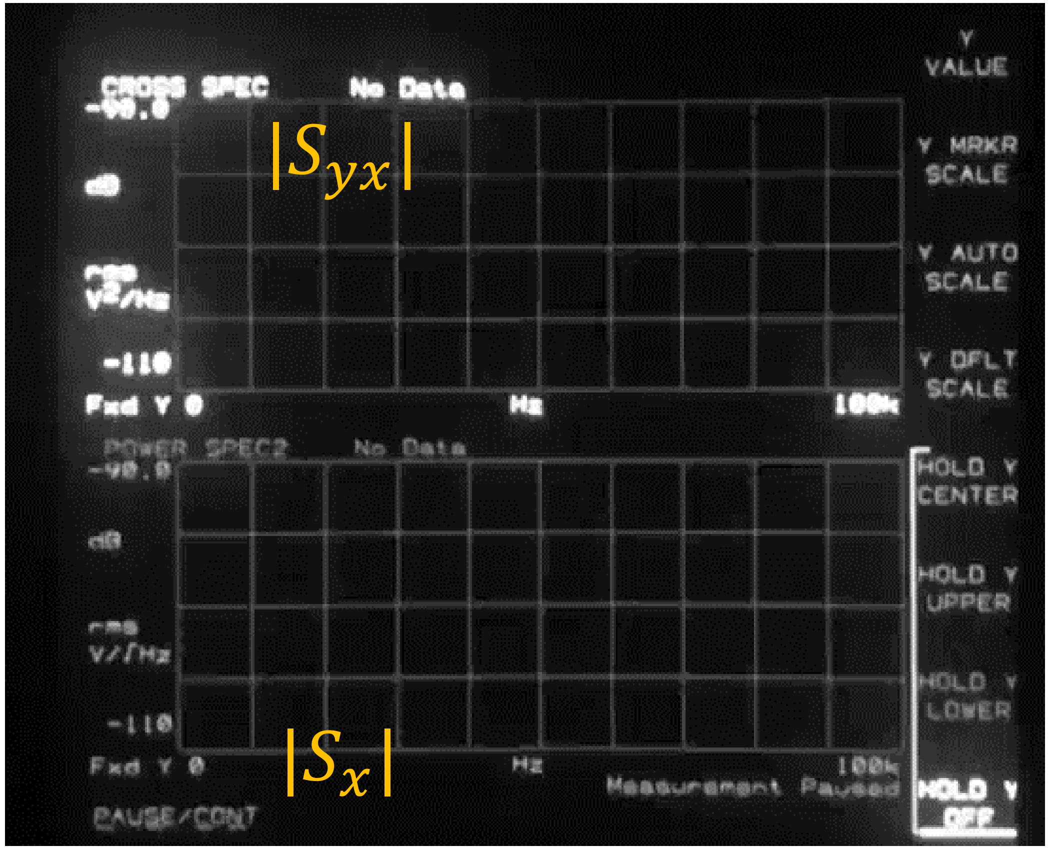
	single ch.	cross	
input	4.6	2.25	nV/√Hz
output	5.6	11.5	μV/√Hz

Experiment – Noise of a 305 Ω resistor

Estimator $\widehat{S_{yx}} = \Re\{S_{yx}\}$, and $\Im\{S_{yx}\}$



Estimator $\widehat{S_{yx}} = |S_{yx}|$, and $|S_x|$



Focus on $\mathbb{E}$ and $\mathbb{V}$

	Term	$\mathbb{E}$	$\mathbb{V}$	PDF	Note
$\mathfrak{R}$	$\langle B'A' + B''A'' + B'C' + B''C'' + C'A' + C''A''\rangle_m$ Bessel K_0, $\mu = 0,\ \sigma^2 = \kappa^2/4$ Bessel K_0, $\mu = 0,\ \sigma^2 = \kappa^2/4$	0	$\frac{1+2\kappa^2}{2m}$	Gauss	average of zero-mean Gaussian processes
$\mathfrak{I}$	$\langle B''A' + B'A'' + B''C' + B'C'' + C''A' + C'A''\rangle_m$	0	$\frac{1+2\kappa^2}{2m}$	Gauss	average of zero-mean Gaussian processes
$\mathfrak{R}$	$\langle C'^2 + C''^2\rangle_m$ white, χ^2, 2 DF $\mu = \kappa^2,\ \sigma^2 = \kappa^4$	κ^2	κ^4/m	χ^2 $r = 2m$	average of χ^2 processes

Normalization: in 1 Hz bandwidth $\mathbb{V}\{A\} = \mathbb{V}\{B\} = 1,\ \mathbb{V}\{C\} = \kappa^2$
$\mathbb{V}\{A'\} = \mathbb{V}\{A''\} = \mathbb{V}\{B'\} = \mathbb{V}\{B''\} = 1/2,$ and $\mathbb{V}\{C'\} = \mathbb{V}\{C''\} = \kappa^2/2$

Estimator $\hat{S}_{yx} = \Re\{\langle S_{yx}\rangle_m\}$

Best (unbiased) estimator

$$\frac{T}{2}\Re\{\langle S_{yx}\rangle_m\} = \langle B'A' + B''A'' + B'C' + B''C'' + C'A' + C''A''\rangle_m + \langle C'^2 + C''^2\rangle_m$$

$$\mathbb{E} = 0,\ \mathbb{V} = (1+2\kappa^2)/(2m)$$

Noise

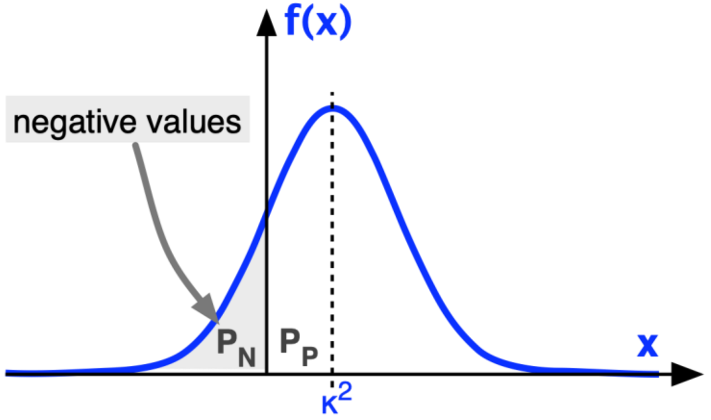
$$\mathbb{E} = \kappa^2,\ \mathbb{V} = \kappa^4/m$$

Signal

$$\mathbb{E}\{\} = \kappa^2$$

$$\sqrt{\mathbb{V}\{\}} = \sqrt{\frac{1+2\kappa^2+2\kappa^4}{2m}} \simeq \frac{1+\kappa^2}{\sqrt{2m}}$$

$$\boxed{\frac{\sqrt{\mathbb{V}\{\}}}{\mathbb{E}}} = \frac{\sqrt{1+2\kappa^2+2\kappa^4}}{\kappa^2\sqrt{2m}} \simeq \boxed{\frac{1+\kappa^2}{\kappa^2\sqrt{2m}}}$$



$$P_N = \mathbb{P}\{\mathbf{x} < 0\} = \frac{1}{2}\mathrm{erfc}\left(\frac{\kappa^2}{\sqrt{2}\,\sigma}\right)$$

0 dB SNR requires that $m = 1/2\kappa^4$.
Example $\kappa = 0.1$ (DUT noise 20 dB lower than single-channel background).
Averaging on 5×10^3 spectra is necessary to get SNR = 0 dB.

Estimator $\hat{S}_{yx} = |\langle S_{yx}\rangle_m|\,,\ \kappa \to 0$

The default of most instruments

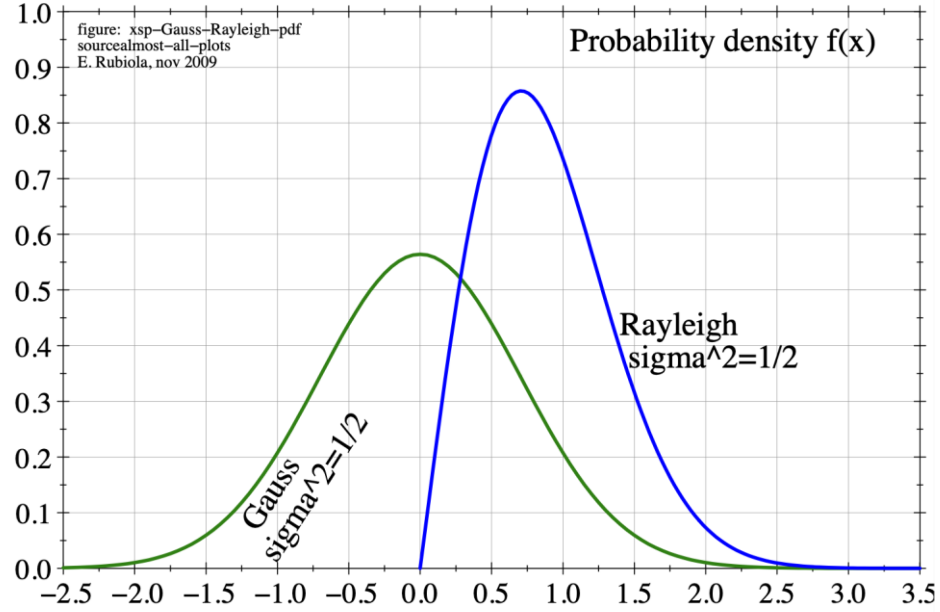
$$|\langle S_{yx}\rangle_m| = \frac{2}{T}\sqrt{[\Re\{\langle \underset{\text{noise, Re}}{Y} \underset{\text{signal}}{X^*}\rangle_m\}]^2 + [\Im\{\langle Y \underset{\text{noise, Im}}{X^*}\rangle_m\}]^2}$$

$\kappa \to 0$ Rayleigh distribution

$$\tfrac{T}{2}\mathbb{E}\left\{\left|\langle S_{yx}\rangle_m\right|\right\} = \sqrt{\frac{\pi}{4m}} = \frac{0.886}{\sqrt{m}}$$

$$\tfrac{T}{2}\mathbb{V}\left\{\left|\langle S_{yx}\rangle_m\right|\right\} = \frac{1}{m}\left(1-\frac{\pi}{4}\right) = \frac{0.215}{m}$$

$$\frac{\operatorname{dev}\left\{\left|\langle S_{yx}\rangle_m\right|\right\}}{\mathbb{E}\left\{\left|\langle S_{yx}\rangle_m\right|\right\}} = \sqrt{\frac{4}{\pi}-1} = 0.523$$

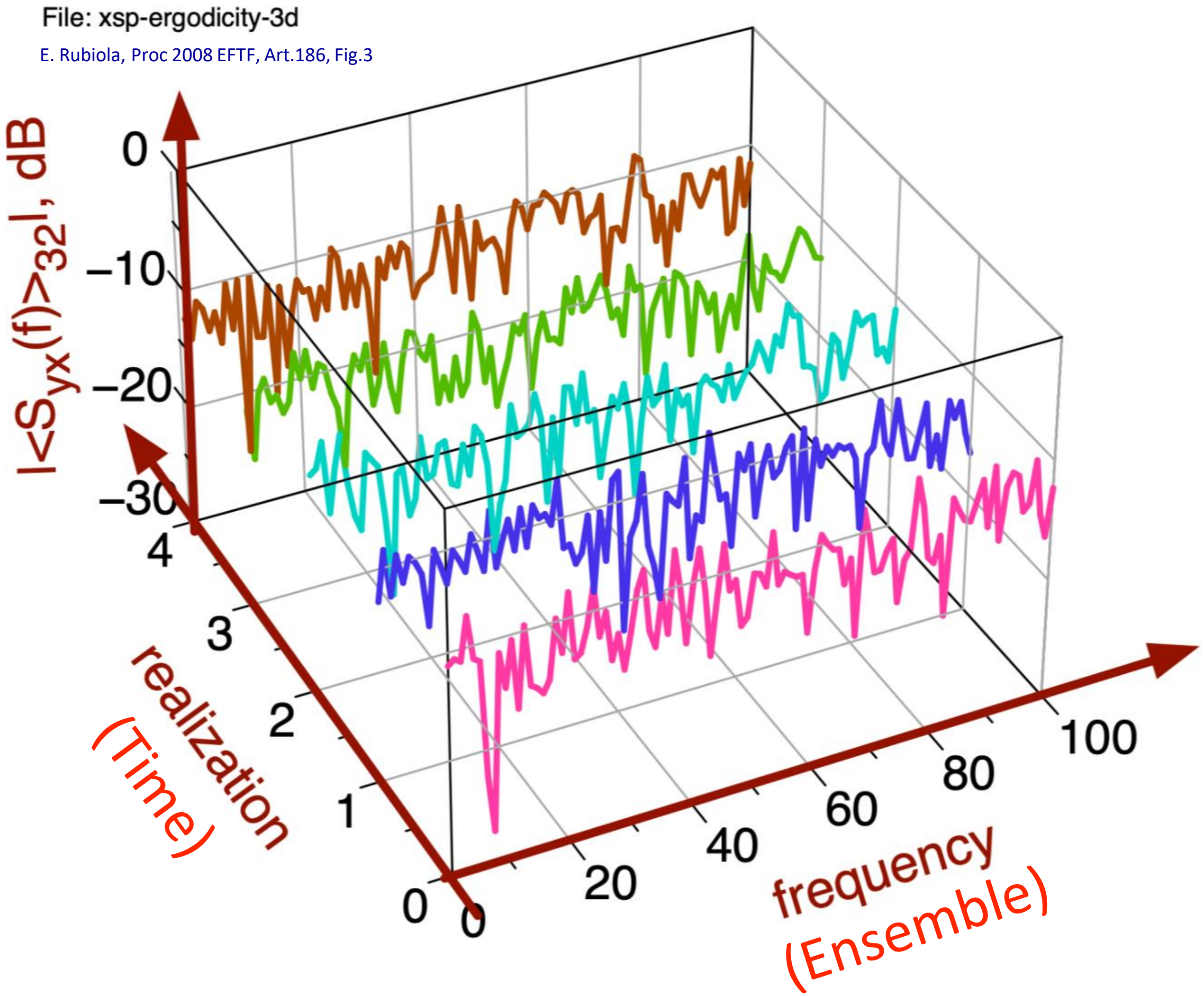


Normalization: in 1 Hz bandwidth $\mathbb{V}\{A\} = \mathbb{V}\{B\} = 1,\ \mathbb{V}\{C\} = \kappa^2$
$\mathbb{V}\{A'\} = \mathbb{V}\{A''\} = \mathbb{V}\{B'\} = \mathbb{V}\{B''\} = 1/2,$ and $\mathbb{V}\{C'\} = \mathbb{V}\{C''\} = \kappa^2/2$

Ergodicity

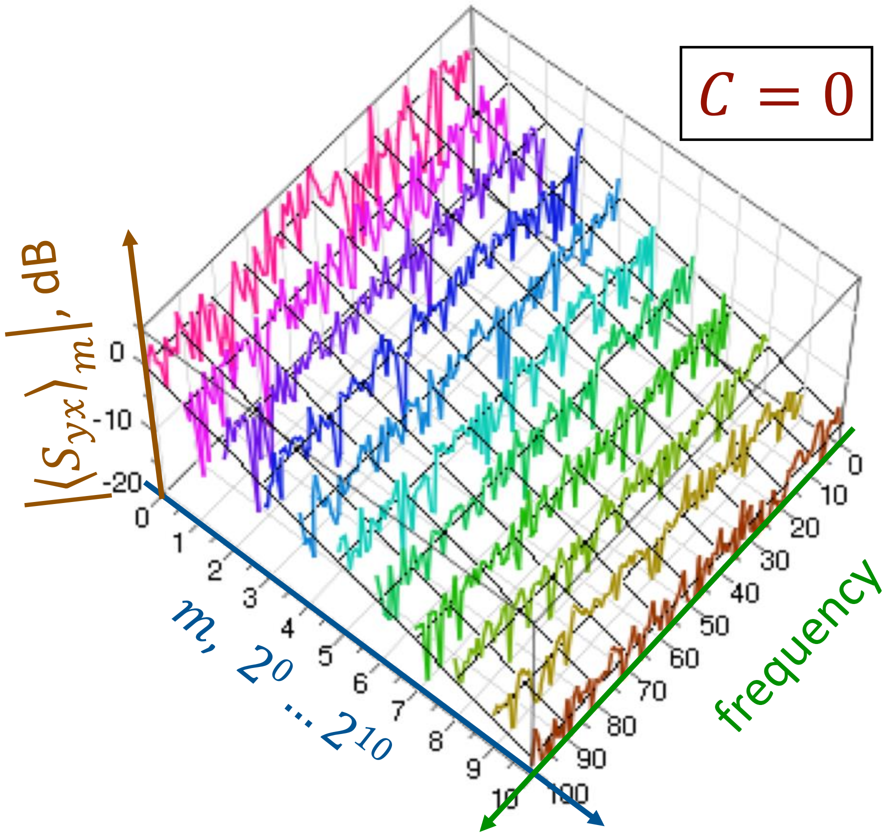
Let’s collect a sequence of spectra

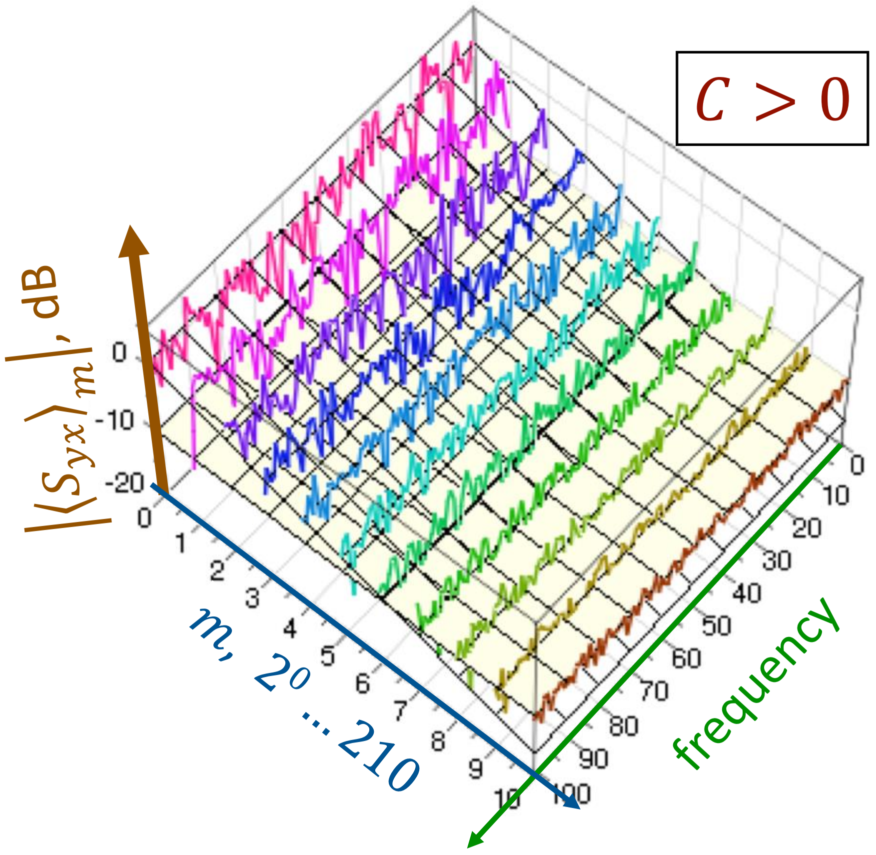
- Ergodicity —> Interchange
  - time /ensemble statistics
  - sequence-index i and frequency f.
- Same average and the deviation on
  - frequency axis
  - sequence of spectra



E. Rubiola, The Magic of Cross Correlation in Measurements from DC to Optics, Proc EFTF, Art n.186, April 2008
E. Rubiola, F. Vernotte, The cross-spectrum experimental method, arXiv:1003.0113 [physics.ins-det], 27 Feb 2010

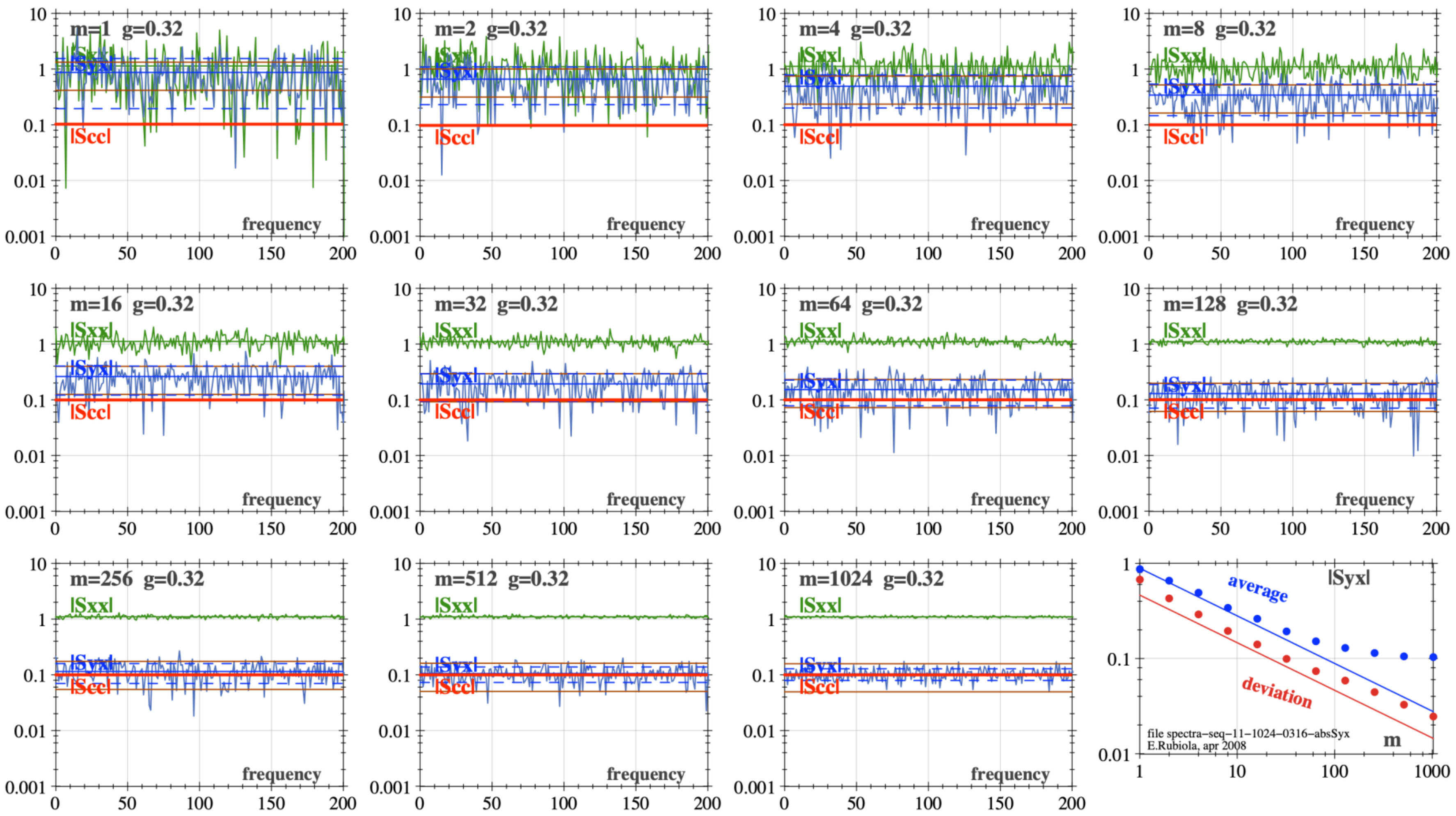
Example: $|S_{yx}|$


$C = 0$
$\left|\langle S_{yx}\rangle_m\right|$, dB
0
-10
-20
0
1
2
3
4
5
6
7
8
9
10
m, $2^0 \ldots 2^{10}$
0
10
20
30
40
50
60
70
80
90
100
frequency


$C > 0$
$\left|\langle S_{yx}\rangle_m\right|$, dB
0
-10
-20
0
1
2
3
4
5
6
7
8
9
10
m, $2^0 \ldots 210$
0
10
20
30
40
50
60
70
80
90
100
frequency

Measurement of $|S_{yx}|$ with $\kappa > 0$

video -> skip

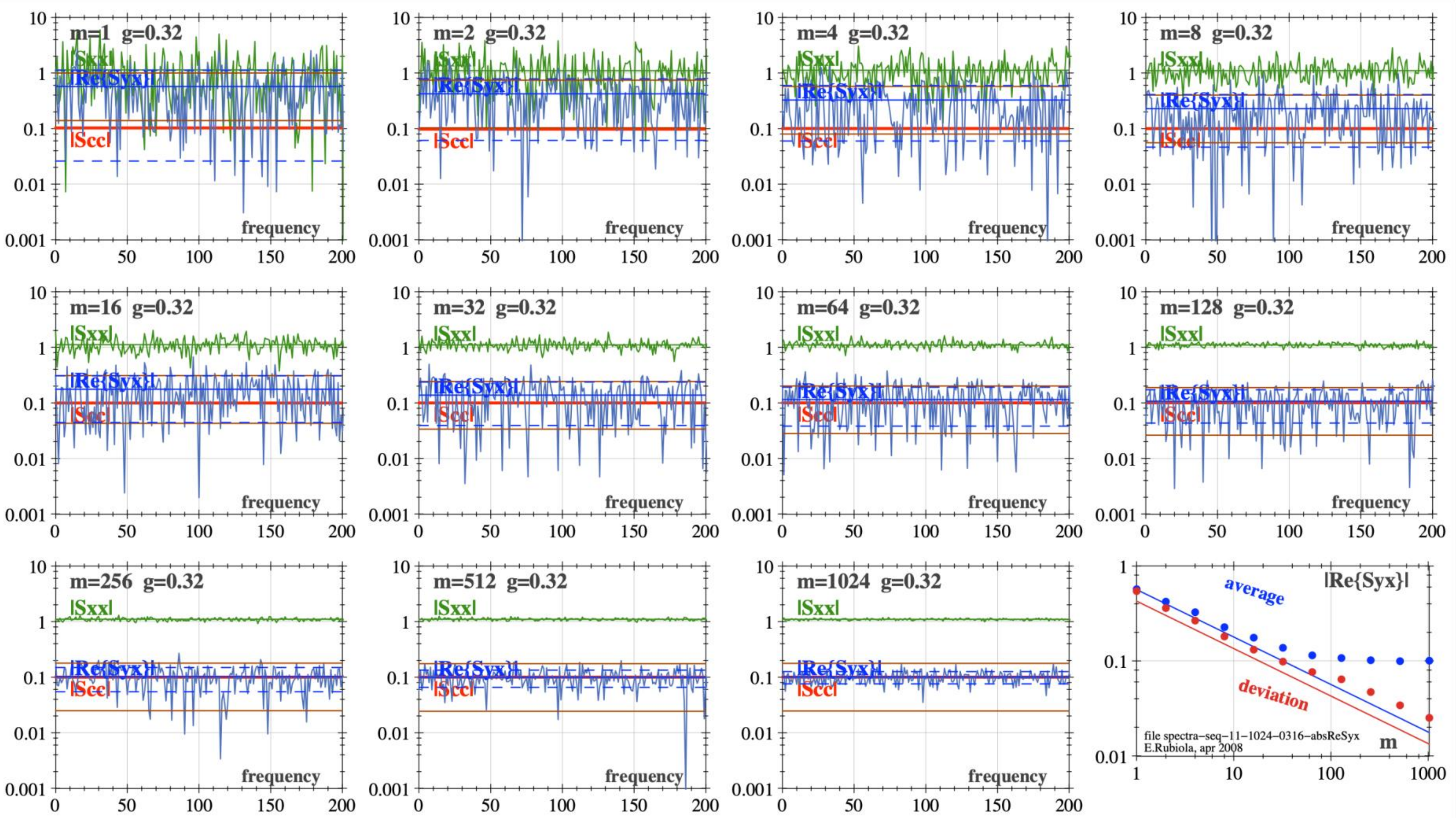


Running the measurement, m increases

S_{xx} shrinks => better confidence level

S_{yx} decreases => higher single-channel noise rejection

Measurement of $\Re\{S_{yx}\}$ with $\kappa > 0$



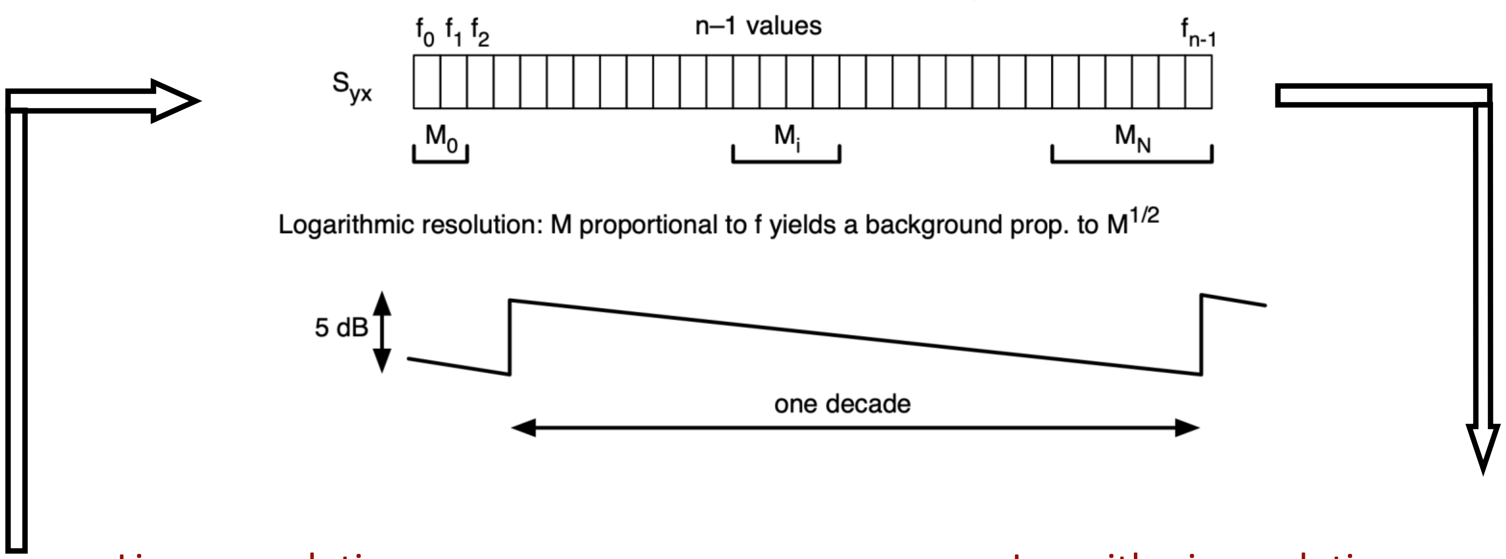
Running the measurement, m increases

S_{xx} shrinks => better confidence level

S_{yx} decreases => higher single-channel noise rejection

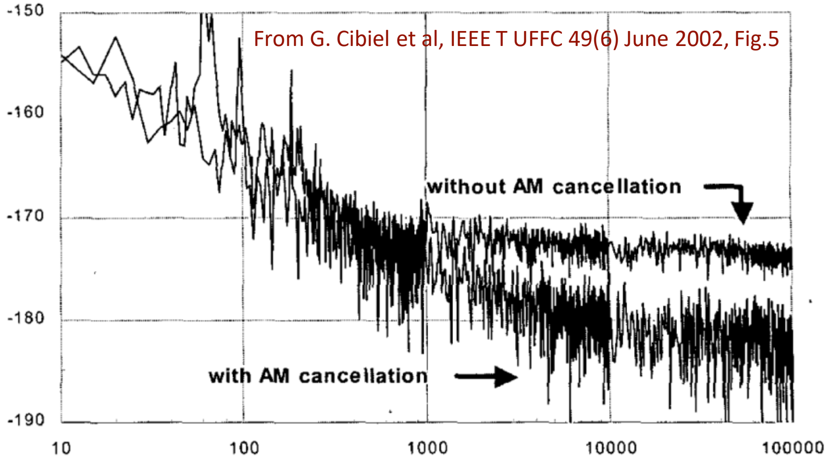
Linear vs logarithmic resolution

Joining M values => background reduction of $M^{1/2}$ because $S(f_j)$, $S(f_k)$, j≠k are independent



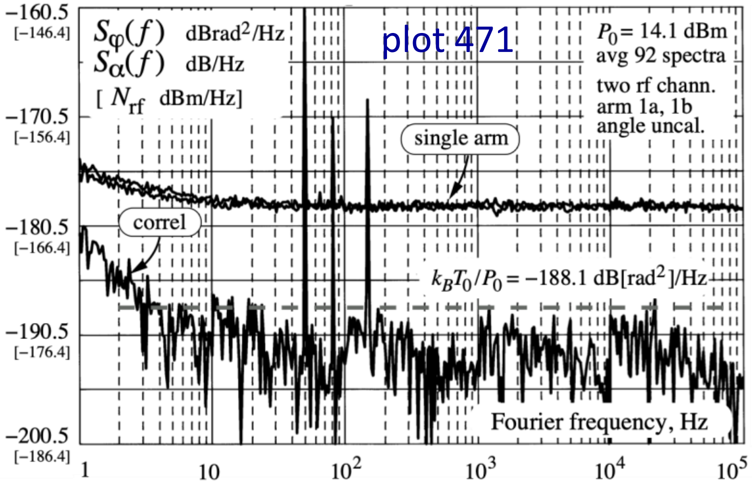
Logarithmic resolution: M proportional to f yields a background prop. to $M^{1/2}$

Linear resolution



From G. Cibiel et al, IEEE T UFFC 49(6) June 2002, Fig.5

Logarithmic resolution



plot 471

From E. Rubiola, V. Giordano, RSI 73(6) jun 2002, Fig.7

Conclusions

- Rejection of the instrument noise
- AM noise, RIN, etc. –> validation of the instrument without a reference low-noise source
- Display quantities

  $\langle \Re\{S_{yx}\} \rangle_m$ is the best estimator, fast and accurate

  $\langle \Im\{S_{yx}\} \rangle_m$ gives the background noise

  $\left| \langle S_{yx} \rangle_m \right|$ is a poor choice: biased, and 4-fold measurement time
- Applications in many fields of metrology

The cross spectrum method is magic

Correlated noise makes magic difficult

home page http://rubiola.org

Appendix: Statistics

Boring but necessary exercises

Vocabulary of statistics

- A random process $\mathbf{x}(t)$ is defined through a random experiment e that associates a function $x_{\mathrm{e}}(t)$ to each outcome e.
  - The set of all the possible $x_{\mathrm{e}}(t)$ is called ensemble
  - The function $x_{\mathrm{e}}(t)$ is called realization or sample function.
  - The ensemble average is called mathematical expectation $\mathbb{E}\{\}$
- A random process is said stationary if its statistical properties are independent of time.
  - Often we restrict the attention to some statistical properties.
  - Broadly similar to the physical concept of repeatability.
- A random process $\mathbf{x}(t)$ said ergodic if a realization observed in time has the statistical properties of the ensemble.
  - Ergodicity makes sense only for stationary processes.
  - Often we restrict the attention to some statistical properties.
  - Broadly similar to the physical concept of reproducibility

Example: thermal noise of a resistor of value R

- The experiment e is the random choice of a resistor e
- The realization $x_{\mathrm{e}}(t)$ is the noise waveform measured across the resistor e
- We always measure $\langle x^2 \rangle = 4kTR\Delta f$, so the process is stationary
- After measuring many resistors, we conclude that $\langle x^2 \rangle = 4kTR\Delta f$ always holds. The process is ergodic.

A relevant property of noise

A theorem states that

there is no a-priori relation
between PDF[1] and PSD

For example, white noise can originate from

- Poisson process (emission of a particle at random time)
- Random telegraph (random switch between two level)
- Thermal noise (Gaussian)

(1) PDF = Probability Density Function

Why white Gaussian noise?

- Whenever randomness occurs at microscopic level, noise tends to be Gaussian (central-limit theorem)

- Most environmental effects are not “noise” in strict sense (often, they are more *disturbing* than noise)

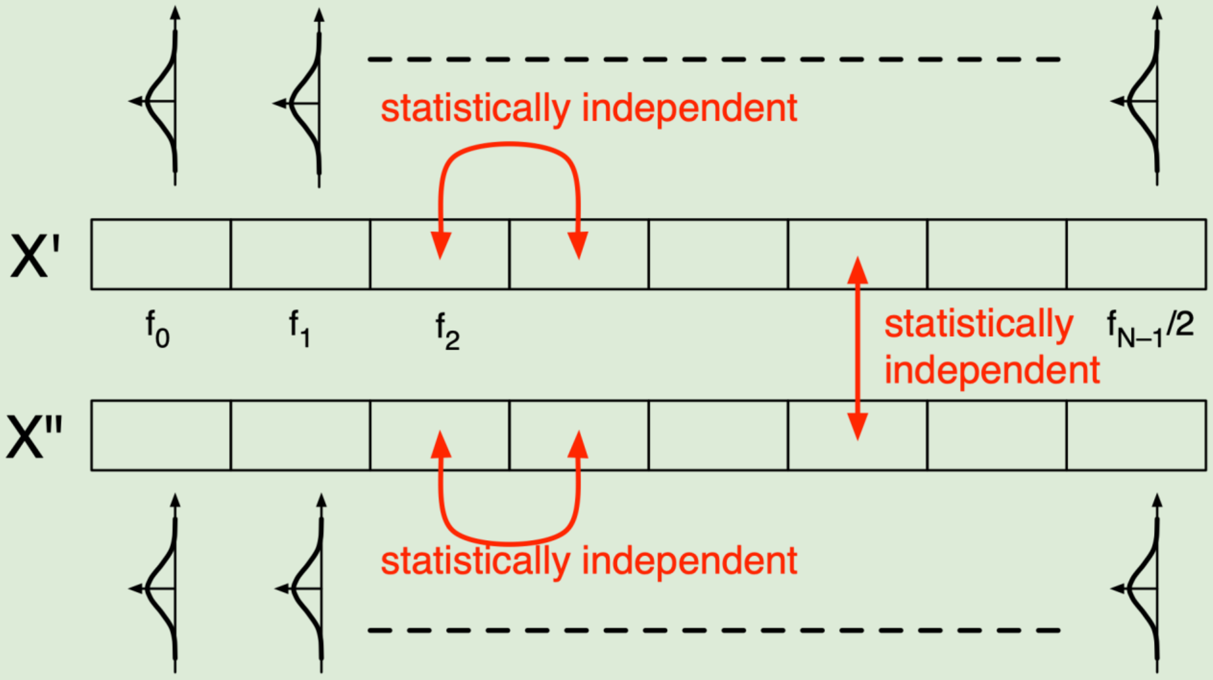
- Colored noise types ($1/f$, $1/f^2$, etc.) can be whitened, analyzed, and un-whitened

- Of course, WG noise is easy to understand

Zero-mean white Gaussian noise

$$x(t) \leftrightarrow X(f) = X'(f) + iX''(f)$$

1. Both $x(t) \leftrightarrow X(f)$ are Gaussian
2. $X(f_1)$ and $X(f_2)$, $f_1 \neq f_2$
   1. are statistically independent,
   2. $\mathbb{V}\{X(f_1)\} = \mathbb{V}\{X(f_2)\}$
3. real and imaginary part:
   1. X' and X'' are statistically independent
   2. $\mathbb{V}\{X'\} = \mathbb{V}\{X''\} = \frac{1}{2}\mathbb{V}\{X\}$
4. $Y = X_1 + X_2$
   1. Y is Gaussian
   2. $\mathbb{V}\{Y\} = \mathbb{V}\{X_1\} + \mathbb{V}\{X_2\}$
5. $Y = X_1 X_2$
   1. Y is Bessel K_0
   2. $\mathbb{V}\{Y\} = \mathbb{V}\{X_1\}\mathbb{V}\{X_2\}$

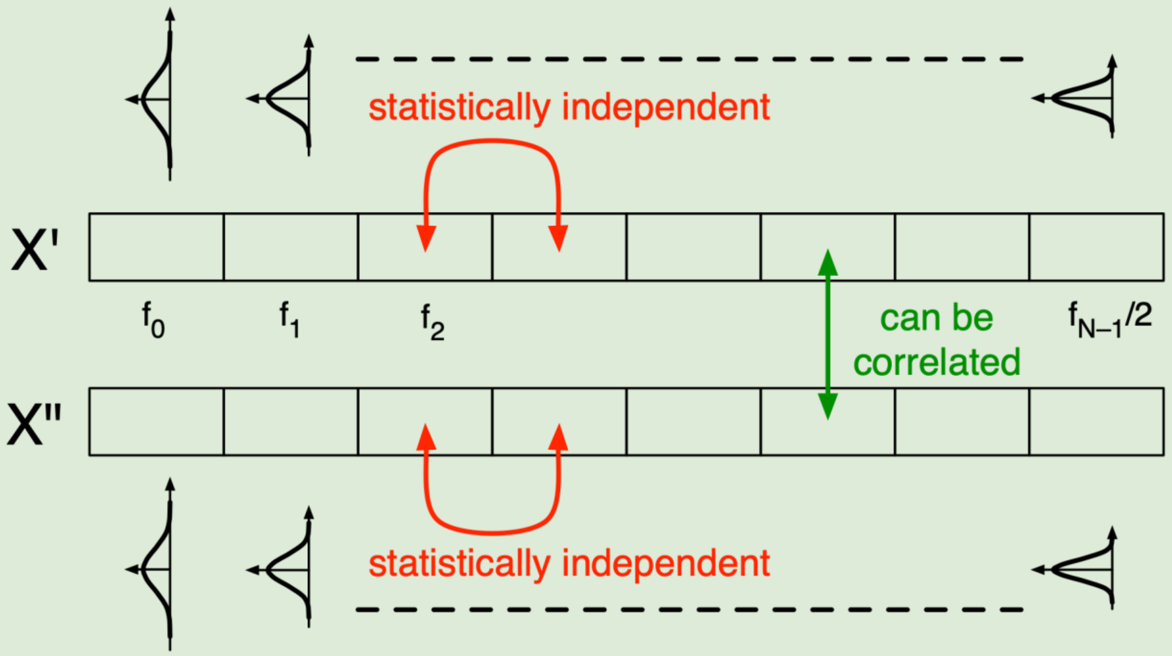


N degrees of freedom

Properties of parametric noise

$$x(t) \leftrightarrow X(f) = X'(f) + iX''(f)$$

1. Pair $x(t) \leftrightarrow X(f)$
   1. there is no a-priori relation between the distribution of $x(t)$ and $X(f)$ (theorem)
   2. Central limit theorem: $x(t)$ and $X(f)$ end up to be Gaussian
2. $X(f_1)$ and $X(f_2)$
   1. generally, statistically independent
   2. $\mathbb{V}\{X(f_1)\} \neq \mathbb{V}\{X(f_2)\}$ in general
3. Real and imaginary part, same frequency
   1. $X'(f)$ and $X''(f)$ can be correlated
   2. in general, $\mathbb{V}\{X'\} \neq \mathbb{V}\{X''\}$
4. $Y = X_1 + X_2$, zero-mean independent Gaussian
   $$\mathbb{V}\{Y\} = \mathbb{V}\{X_1\} + \mathbb{V}\{X_2\}$$
5. If X_1 and X_2 are zero-mean independent Gaussian
   1. $Y = X_1 X_2$ is zero-mean Bessel K
   2. $\mathbb{V}\{Y\} = \mathbb{V}\{X_1\}\mathbb{V}\{X_2\}$



N degrees of freedom

Gaussian (normal) distribution

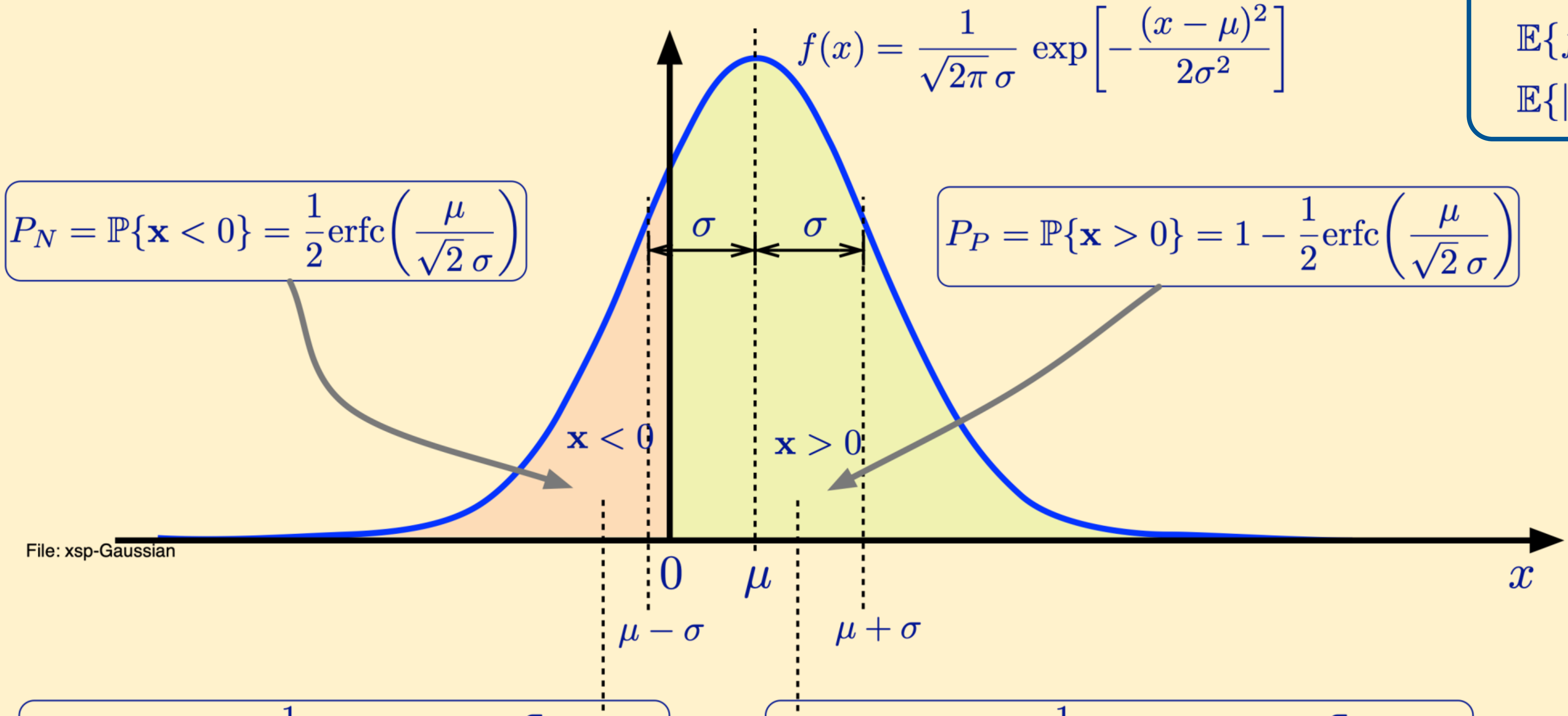
x is normal distributed with mean μ and variance σ^2

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\mathbb{E}\{f(x)\} = \mu$$

$$\mathbb{E}\{f^2(x)\} = \mu^2 + \sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$



$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$P_N = \mathbb{P}\{\mathbf{x} < 0\} = \frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)$$

$$P_P = \mathbb{P}\{\mathbf{x} > 0\} = 1 - \frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)$$

$$\mu_N = \mu - \frac{1}{\frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)} \frac{\sigma}{\sqrt{2\pi\exp(\mu^2/\sigma^2)}}$$

$$\mu_P = \mu + \frac{1}{1 - \frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sqrt{2}\,\sigma}\right)} \frac{\sigma}{\sqrt{2\pi\exp(\mu^2/\sigma^2)}}$$

Sum and average of random variables

1. The central limit theorem states that

   For large m, the PDF of the sum of m statistically independent processes tends to a Gaussian distribution

2. Let $X = X_1 + X_2 + \cdots + X_m$ be the sum of m processes of mean $\mu_1, \mu_2 \ldots \mu_m$ and variance $\sigma_1^2, \sigma_2^2, \ldots \sigma_m^2$. The process X tends to Gaussian PDF, expectation

   Expectation $\mathbb{E}\{X\} = \mu_1 + \mu_2 + \cdots + \mu_m$

   Variance $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2$

3. The average $\langle X \rangle_m = \frac{1}{m}(X_1 + X_2 + \cdots + X_m)$ has Gaussian PDF,

$$\mathbb{E}\{X\} = \frac{1}{m}(\mu_1 + \mu_2 + \cdots + \mu_m), \text{ and}$$

$$\sigma^2 = \frac{1}{m}(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2)$$

Since white noise and flicker noise arise from the sum of a large number of small-scale phenomena, they are Gaussian distributed

PDF = Probability Density Function

Children of the Gaussian distribution

Chi-square

$$\chi^2 = \sum_i x_i^2$$

Bessel K_0

$$x = x_1 x_2$$

Rayleigh

$$x = \sqrt{x_1^2 + x_2^2}$$

One-Sided Gaussian

Chi-square (χ^2) distribution

Definition

DF = degrees of freedom

x_i are normal distributed variables

zero mean, and variance σ^2

$$\chi^2 = \sum_{i=1}^{r} x_i^2$$

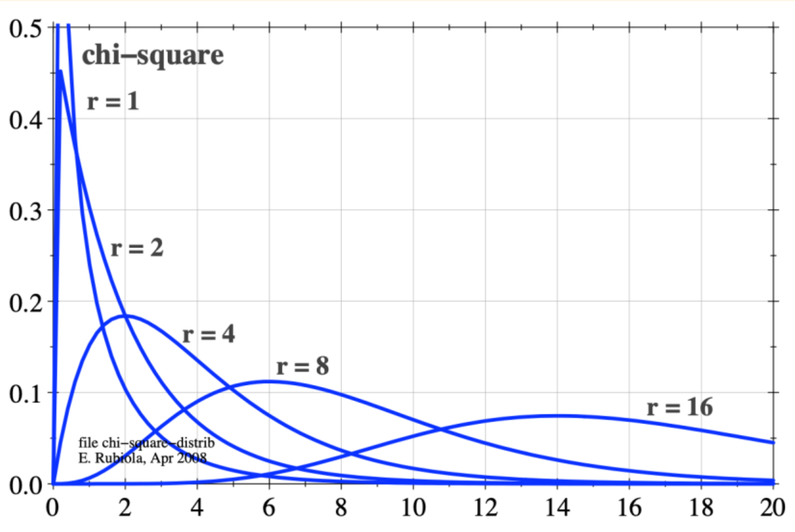
is χ^2 distributed with r DF

Sum

The sum of m χ^2-distributed variables

$$\chi^2 = \sum_{j=1}^{m} \chi_j^2 \;, \qquad r = \sum_{j=1}^{m} r_j$$

has χ^2 distribution with $r = m$ DF



file chi–square–distrib
E. Rubiola, Apr 2008

$$f(x) = \frac{x^{\frac{r}{2}-1}\, e^{-\frac{x^2}{2}}}{\Gamma\left(\frac{1}{2}r\right) 2^{\frac{r}{2}}} \qquad x \geq 0$$

$$\mathbb{E}\{f(x)\} = \sigma^2 r$$

$$\mathbb{E}\{[f(x)]^2\} = \sigma^4 r(r+2)$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = 2\sigma^4 r$$

$$z! = \Gamma(z+1), \quad z \in \mathbb{N}$$

Averaging m complex χ^2 variables

averaging m variables$|X|^2$, complex $X = X' + iX''$,
yields a χ^2 distribution with $r = 2m$

$$\frac{1}{m}\chi^2 = \frac{1}{m}\sum_{j=1}^{m} (X_j')^2 + (X_j'')^2$$

$$\mathbb{E}\left\{\frac{1}{m}f(x)\right\} = \frac{\sigma^2 r}{m} = 2\sigma^2$$

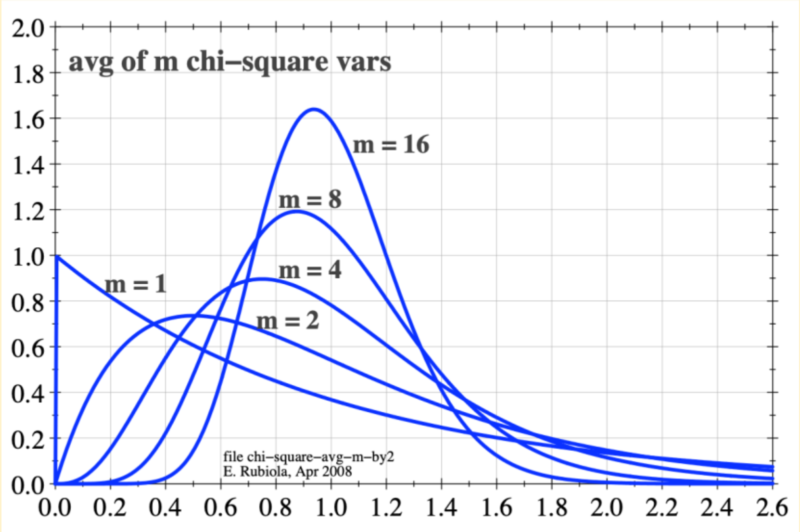
$$\mathbb{E}\left\{\left|\frac{1}{m}f(x) - \mathbb{E}\left\{\frac{1}{m}f(x)\right\}\right|^2\right\} = \frac{2\sigma^4 r}{m^2} = \frac{4\sigma^4}{m}$$

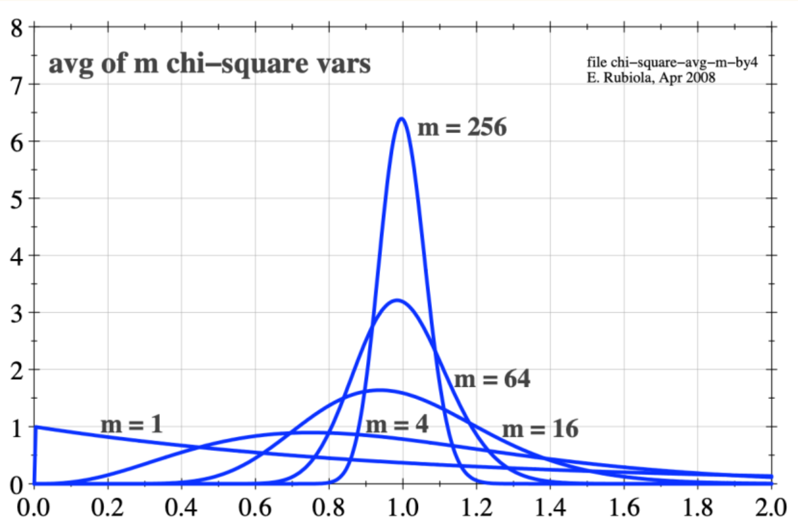
$$\frac{\text{dev}}{\text{avg}} = \frac{1}{\sqrt{m}}$$

relevant case: $\sigma^2 = 1/2$

$\text{avg} = 1$

$\text{dev} = \dfrac{1}{\sqrt{m}}$





Product of independent zero-mean Gaussian random variables

x_1 and x_2 are normal distributed with zero mean and variance σ_1^2, σ_2^2

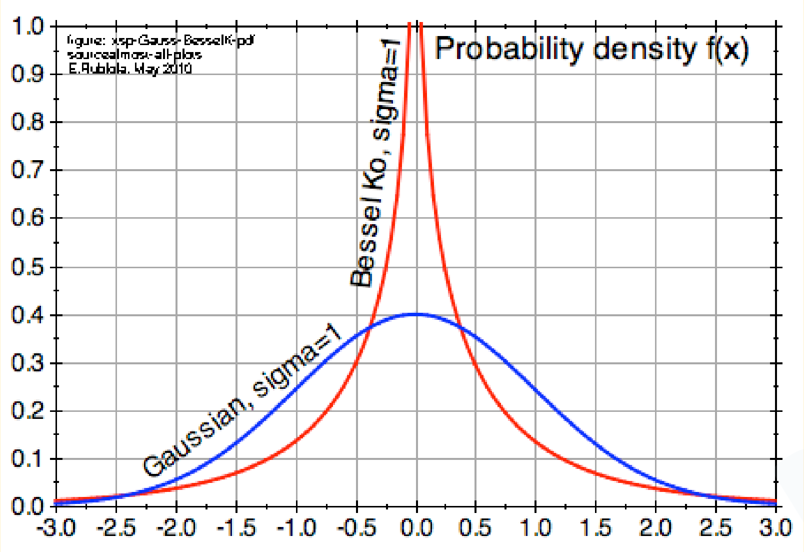
$$x = x_1 x_2$$

x has Bessel K_0 distribution with variance $\sigma^2 = \sigma_1^2 \sigma_2^2$

$$f(x) = \frac{1}{\pi\sigma} K_0 \left(-\frac{|x|}{\sigma} \right)$$

$$\mathbb{E}\{f(x)\} = 0$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$



Bessel K_0 distribution

x_1 and x_2 are normal distributed with zero mean and variance σ_1^2, σ_2^2

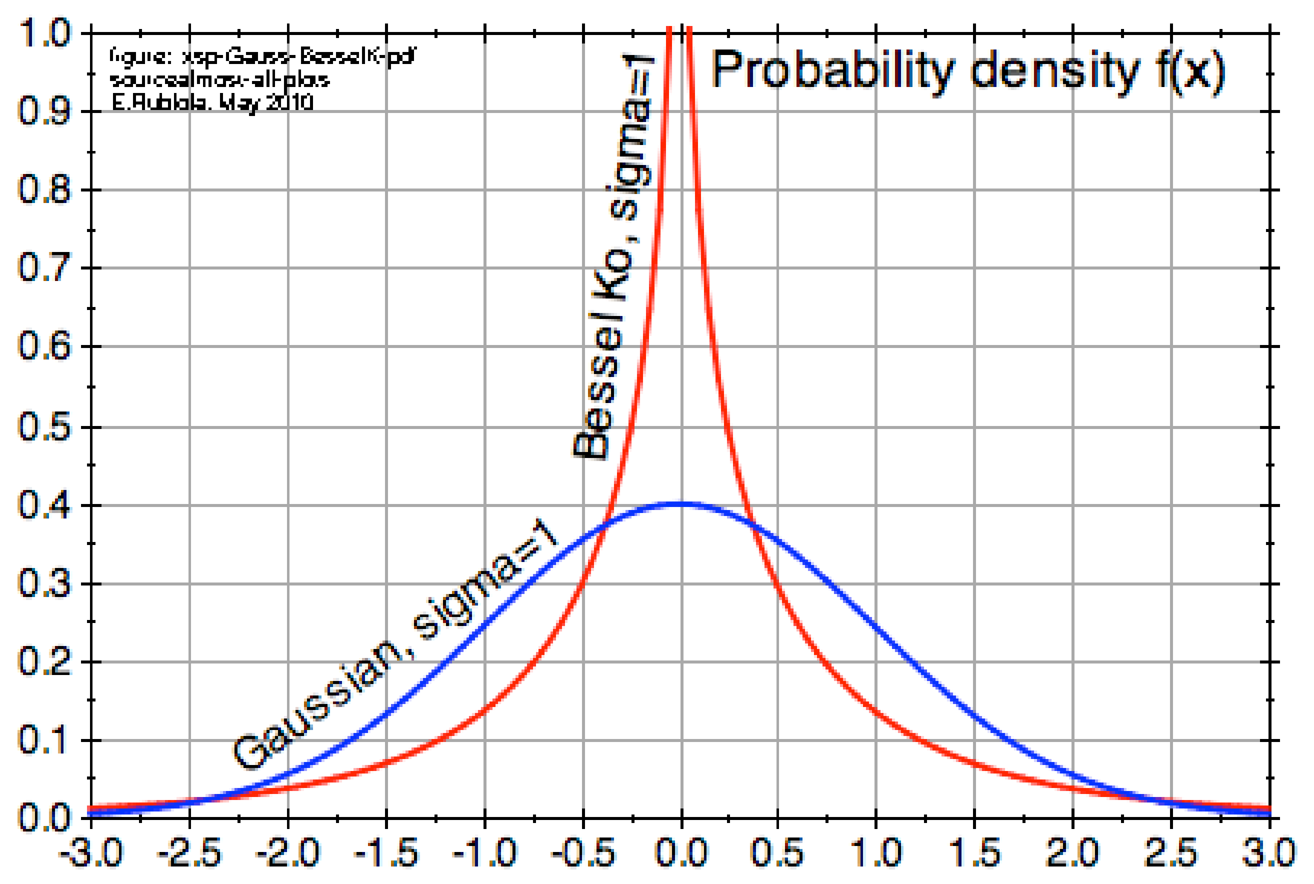
$$x = x_1\, x_2$$

x has Bessel K_0 distribution with variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$

$$f(x) = \frac{1}{\pi\sigma}\, K_0\left(-\frac{|x|}{\sigma}\right)$$

$$\mathbb{E}\{f(x)\} = 0$$

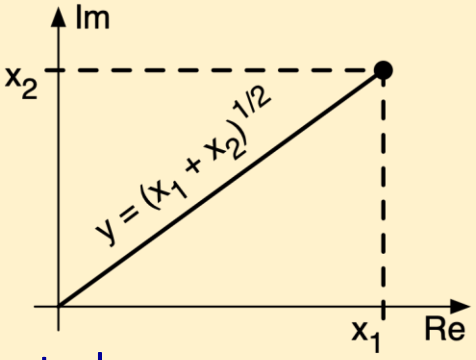
$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \sigma^2$$



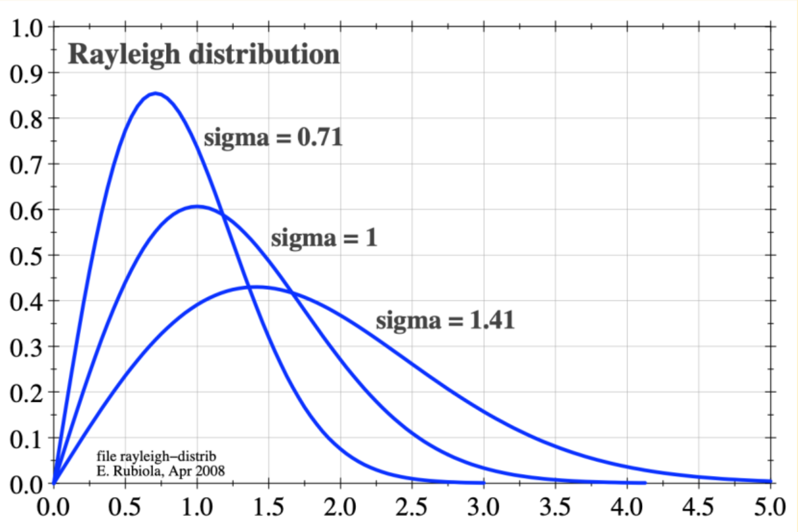
Rayleigh distribution

x_1 and x_2 are normal distributed with zero mean and equal variance σ^2

$$x = \sqrt{x_1^2 + x_2^2}$$



x is Rayleigh-distributed



$$f(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad x \geq 0$$

$$\mathbb{E}\{f(x)\} = \sqrt{\frac{\pi}{2}}\,\sigma$$

$$\mathbb{E}\{f^2(x)\} = 2\sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \frac{4-\pi}{2}\sigma^2$$

Rayleigh distribution with $\sigma^2 = 1/2$

quantity with $\sigma^2 = 1/2$	value [10 log( ), dB]
average $= \sqrt{\frac{\pi}{4}}$	0.886 [−0.525]
deviation $= \sqrt{1 - \frac{\pi}{4}}$	0.463 [−3.34]
$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{4}{\pi} - 1}$	0.523 [−2.82]
$\frac{\text{avg} + \text{dev}}{\text{avg}} = 1 + \sqrt{\frac{4}{\pi} - 1}$	1.523 [+1.83]
$\frac{\text{avg} - \text{dev}}{\text{avg}} = 1 - \sqrt{\frac{4}{\pi} - 1}$	0.477 [−3.21]
$\frac{\text{avg} + \text{dev}}{\text{avg} - \text{dev}} = \frac{1 + \sqrt{4/\pi - 1}}{1 - \sqrt{4/\pi - 1}}$	3.19 [5.04]

One-sided Gaussian distribution

x is normal distributed with zero mean and variance σ^2

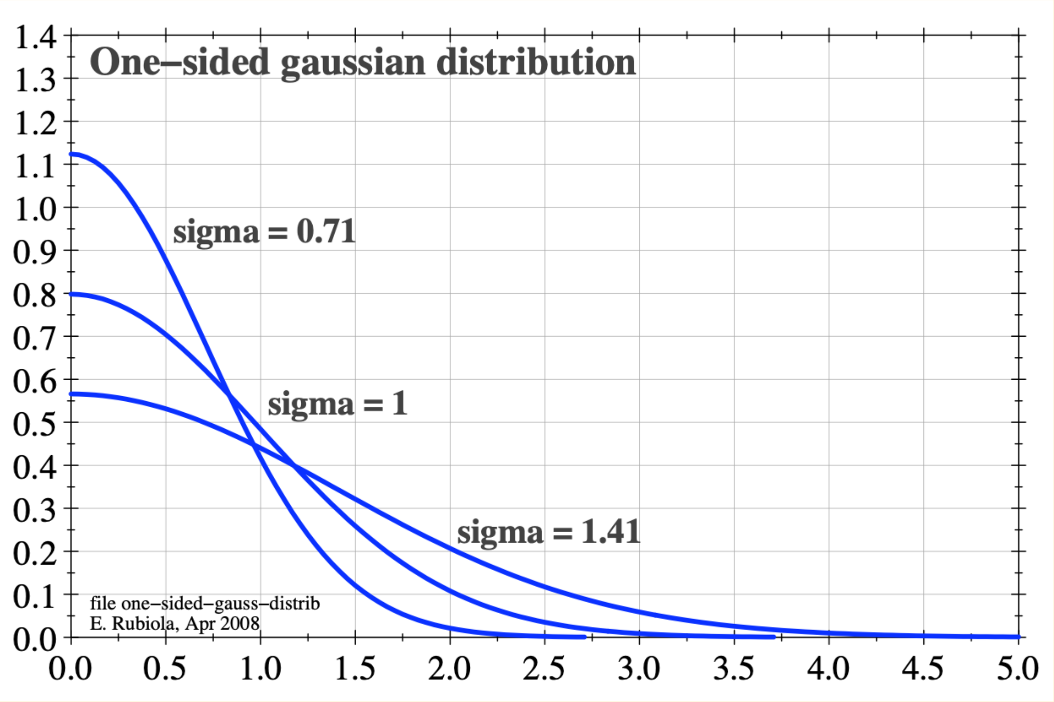
$$y = |x|$$

$$f(x) = 2\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$\mathbb{E}\{f(x)\} = \sqrt{\frac{2}{\pi}}\,\sigma$$

$$\mathbb{E}\{f^2(x)\} = \sigma^2$$

$$\mathbb{E}\{|f(x) - \mathbb{E}\{f(x)\}|^2\} = \left(1 - \frac{2}{\pi}\right)\sigma^2$$



one-sided Gaussian distribution with $\sigma^2 = 1/2$	
quantity with $\sigma^2 = 1/2$	value [10 log( ), dB]
average $= \sqrt{\frac{1}{\pi}}$	0.564 [−2.49]
deviation $= \sqrt{\frac{1}{2} - \frac{1}{\pi}}$	0.426 [−3.70]
$\frac{\text{dev}}{\text{avg}} = \sqrt{\frac{\pi}{2} - 1}$	0.756 [−1.22]
$\frac{\text{avg} + \text{dev}}{\text{avg}} = 1 + \sqrt{\frac{1}{2} - \frac{1}{\pi}}$	1.756 [+2.44]
$\frac{\text{avg} - \text{dev}}{\text{avg}} = 1 - \sqrt{\frac{1}{2} - \frac{1}{\pi}}$	0.244 [−6.12]
$\frac{\text{avg} + \text{dev}}{\text{avg} - \text{dev}} = \frac{1 + \sqrt{1/2 - 1/\pi}}{1 - \sqrt{1/2 - 1/\pi}}$	7.18 [8.56]

Applications of the Cross Spectrum Measurement

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home page http://rubiola.org

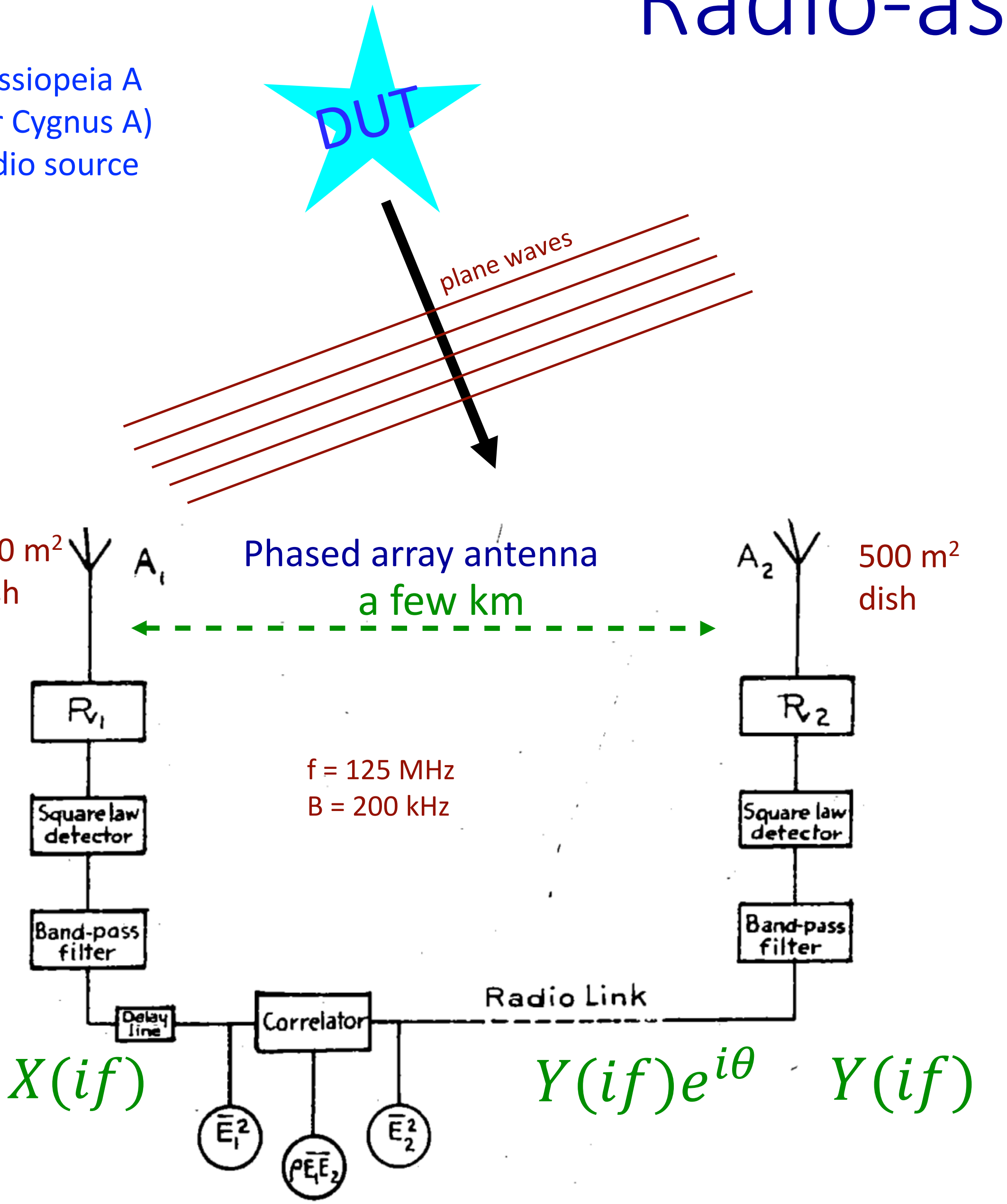
Summary

- Radio-astronomy (Hanbury-Brown, 1952)
- Early implementations
- Radiometry (Allred, 1962)
- Noise calibration (Spietz, 2003)
- Frequency noise (Vessot 1964)
- Phase noise (Walls 1976)
- Dual delay line system (Lance, 1982)
- Phase noise (Rubiola 2000 & 2002)
- Effect of amplitude noise (Rubiola, 2007)
- Frequency stability of a resonator (Rubiola)
- Dual-mixer time-domain instrument (Allan 1975, Stein 1983)
- Amplitude noise & laser RIN (Rubiola 2006)
- Noise of a power detector (Grop & Rubiola)
- Noise in chemical batteries (Walls 195)
- Semiconductors (Sampietro RSI 1999)
- Electromigration in thin films (Stoll 1989)
- Fundamental definition of temperature
- Hanbury Brown - Twiss effect (Hanbury-Brown & Twiss 1956, Glattli 2004)

The real fun starts here

Radio-astronomy

Cassiopeia A
(or Cygnus A)
radio source



500 m^2 dish

Phased array antenna
a few km

500 m^2 dish

f = 125 MHz
B = 200 kHz

$X(if)$ $Y(if)e^{i\theta}$ $Y(if)$

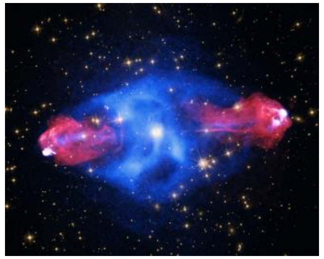
R. Hanbury Brown & al., Nature 170(4338), Fig.1

Measurement of the apparent angular size of stellar radio sources

Jodrell Bank, Manchester, UK

- The radio link breaks the hypothesis of symmetry of the two channels, introducing a phase θ
- The cross spectrum is complex
- The antenna directivity results from the phase relationships
- The phase of the cross spectrum indicates the direction of the radio source

α Cigni (Deneb)



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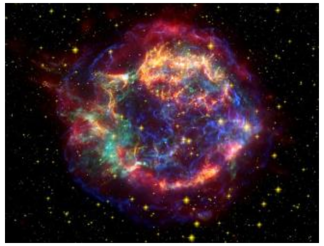
α Cassiopeiae (Schedar)



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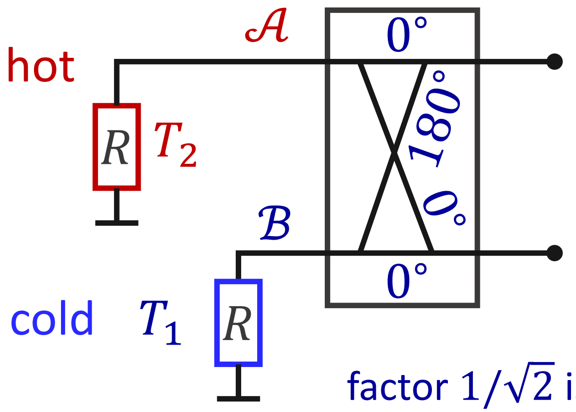
R. Hanbury Brown & al., Nature 170(4338) p.1061-1063, 20 Dec 1952
R. Hanbury Brown, R. Q. Twiss, Phyl. Mag. ser.7 no.366 p.663-682

Radiometry & Johnson thermometry



$X = \frac{1}{\sqrt{2}}(\mathcal{A} + \mathcal{B})$

$Y = \frac{1}{\sqrt{2}}(\mathcal{A} - \mathcal{B})$

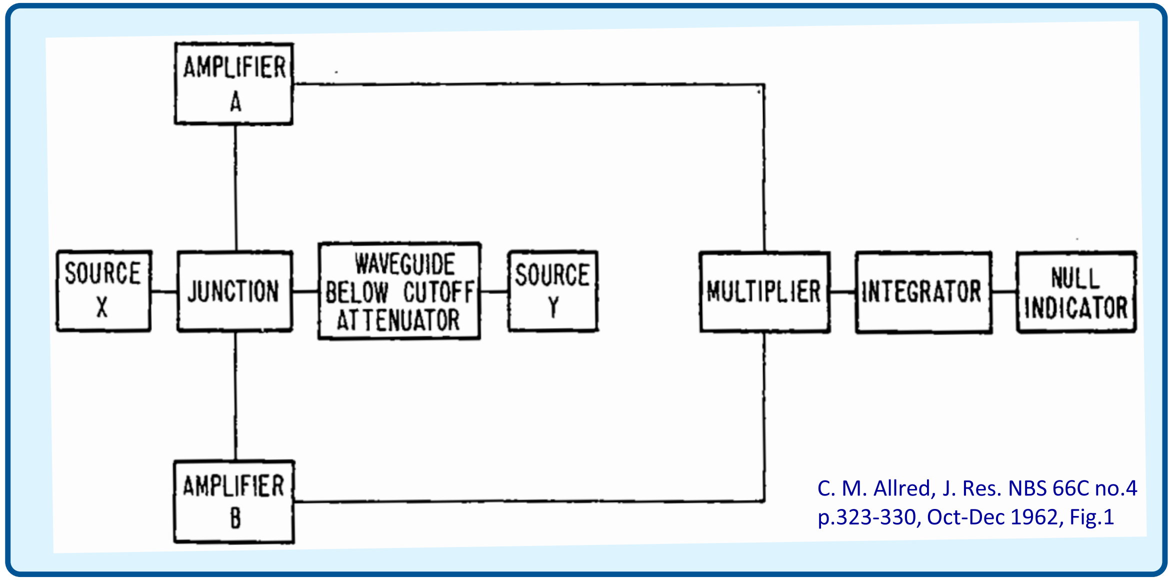
factor $1/\sqrt{2}$ is energy conservation

Temperature difference

$$S_{yx} = \tfrac{1}{2}k(T_2 - T_1)$$

$$T_2 - T_2 < 0 \;\Rightarrow S_{yx} < 0$$

See also E.Rubiola, V.Giordano, RSI 73(6), June 2002



C. M. Allred, J. Res. NBS 66C no.4 p.323-330, Oct-Dec 1962, Fig.1

noise comparator

C. M. Allred, A precision noise spectral density comparator, J. Res. NBS 66C no.4 p.323-330, Oct-Dec 1962

Article made publicly available by NIST, https://nvlpubs.nist.gov/nistpubs/jres/66C/jresv66Cn4p323_A1b.pdf

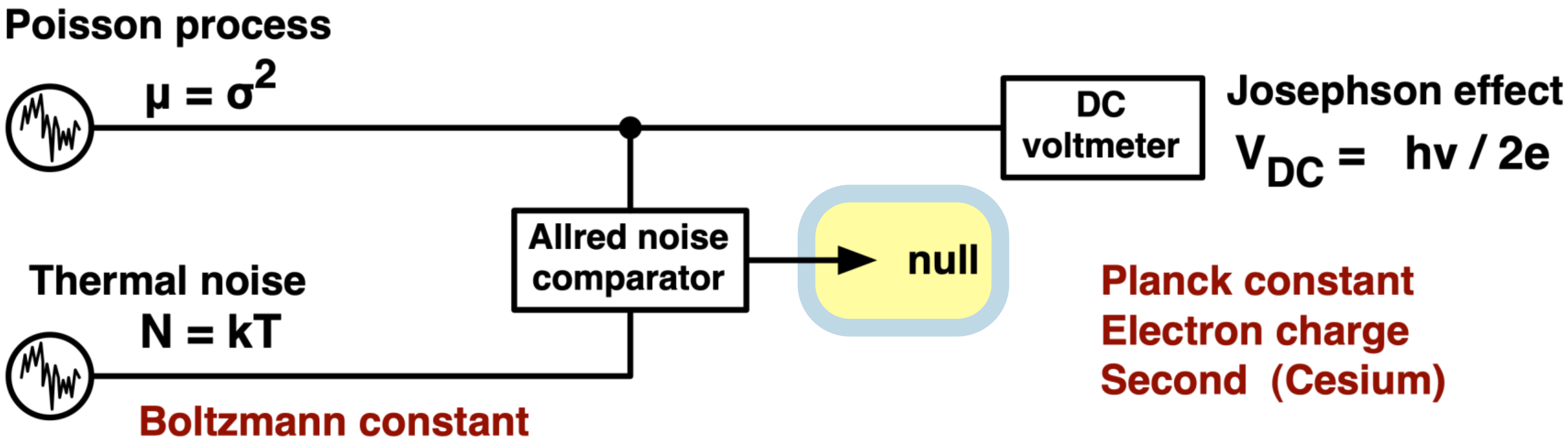
Conceptual implementation of the Kelvin

Boltzmann constant $k = 1.380649 \times 10^{-23}$ J/K exact (≥20 May 2019)

thermal noise $S = kT$

shot noise $S = 2eIR$

high accuracy of I
with a dc instrument



Poisson process

$\mu = \sigma^2$

DC voltmeter

Josephson effect

$V_{DC} = h\nu / 2e$

Allred noise comparator

null

Thermal noise

$N = kT$

Boltzmann constant

Planck constant
Electron charge
Second (Cesium)

Property of the Poisson process

$$\mu = \sigma^2$$

Noise calibration

thermal noise	$S = kT$	high accuracy of I
shot noise	$S = 2eIR$	with a dc instrument

Compare shot and thermal noise with a noise bridge

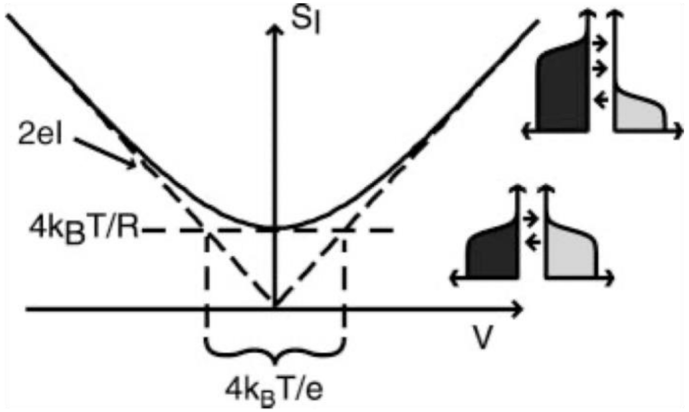


Fig. 1. Theoretical plot of current spectral density of a tunnel junction (Eq. 3) as a function of dc bias voltage. The diagonal dashed lines indicate the shot noise limit, and the horizontal dashed line indicates the Johnson noise limit. The voltage span of the intersection of these limits is $4k_BT/e$ and is indicated by vertical dashed lines. The bottom inset depicts the occupancies of the states in the electrodes in the equilibrium case, and the top inset depicts the out-of-equilibrium case where $eV \gg k_BT$.

L. Spietz & al., Science 300(20) p. 1929-1932, jun 2003

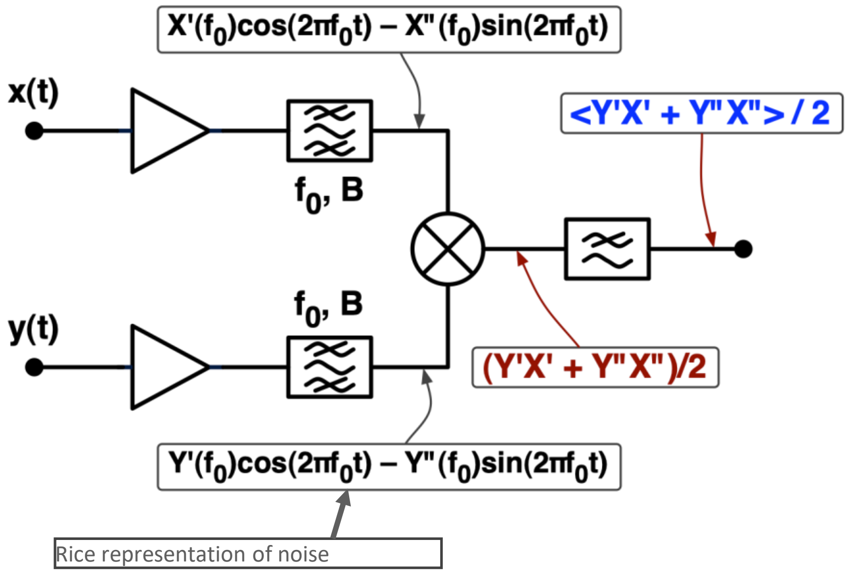
In a tunnel junction, theory predicts the amount of shot and thermal noise

L. Spietz & al., Primary electronic thermometry using the shot noise of a tunnel junction, Science 300(20) p. 1929-1932, jun 2003

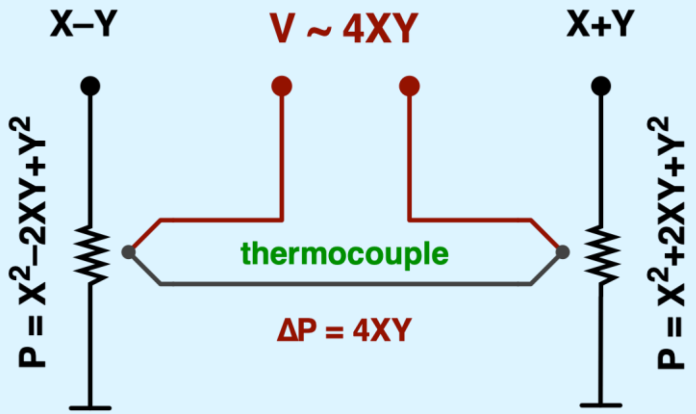
Early implementations

1940-1950 technology

Analog correlator



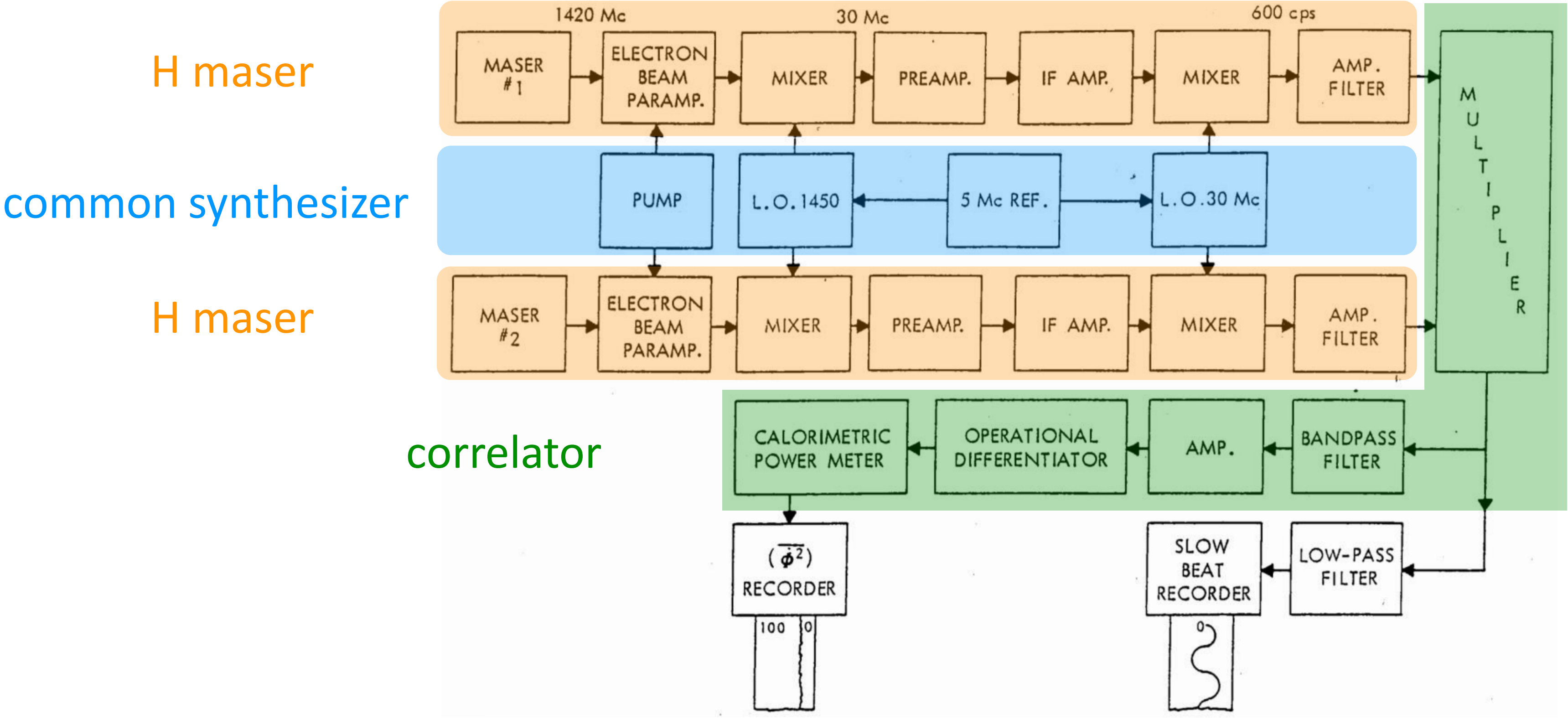
Analog multiplier



Spectral analysis at the single frequency f_0, in the bandwidth B
Need a filter pair for each Fourier frequency

Frequency noise of a H-maser

R. F. C. Vessot & al., Fig.10-4

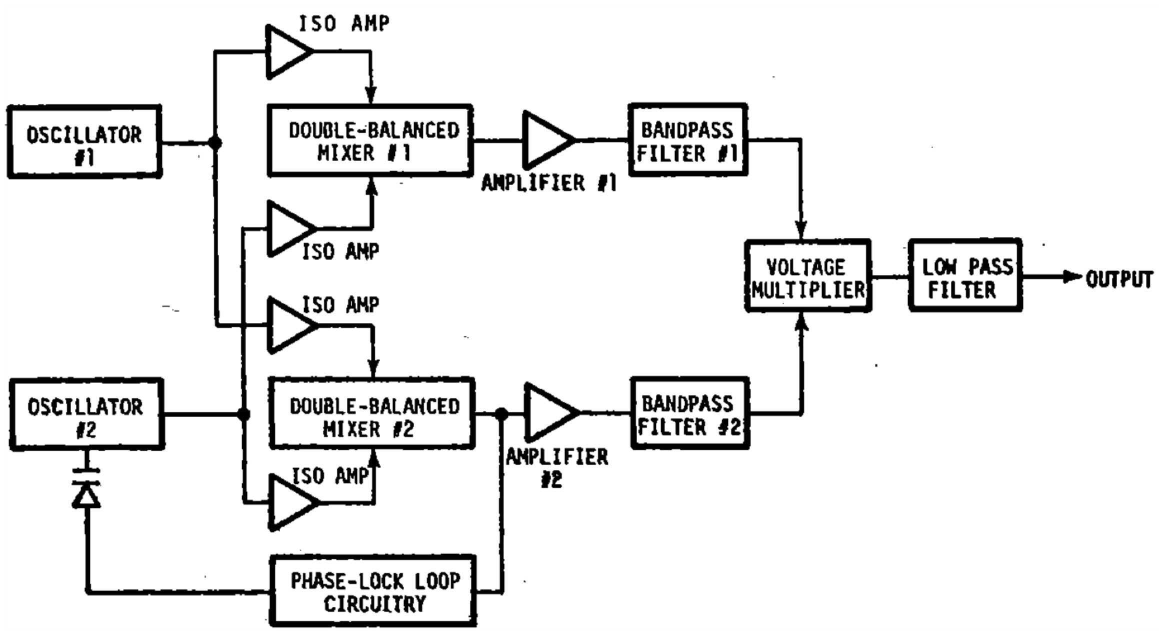


R. F. C. Vessot, L. F. Mueller, J. Vanier, Proc. NASA Symp. on Short Term Frequency Stability p.111-118, Greenbelt, MD, 23-24 Nov 1964

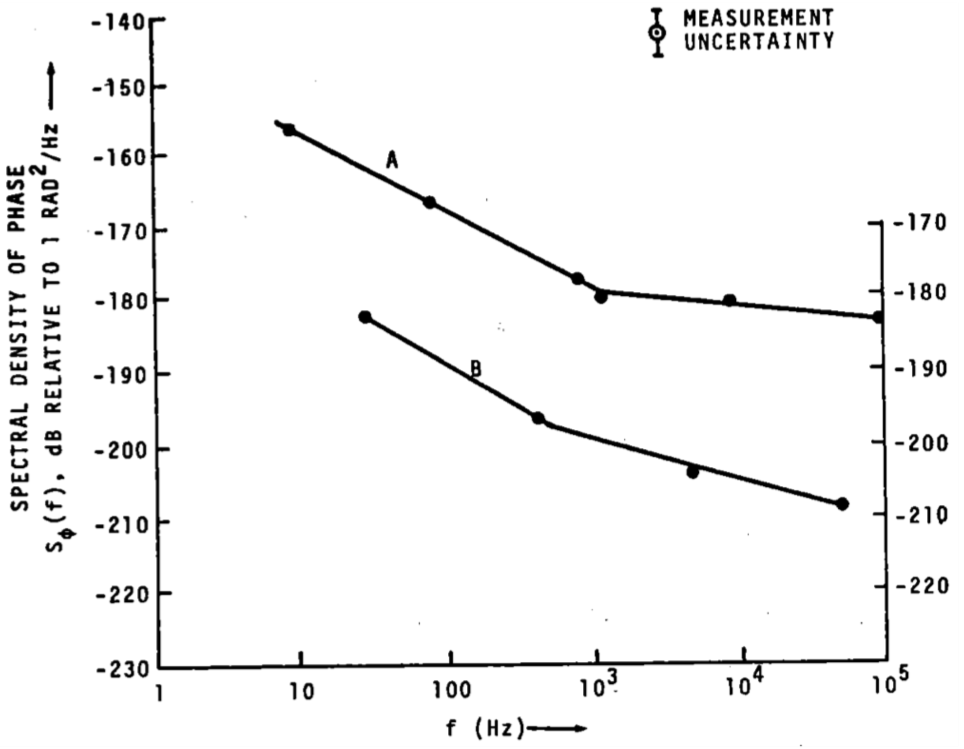
Article made publicly available by NASA https://ntrs.nasa.gov/api/citations/19660001092/downloads/19660001092.pdf

Phase noise measurement

For the lectures on oscillators



F.L. Walls & al, Proc. 30th FCS pp.269-274, 1976, Fig.7



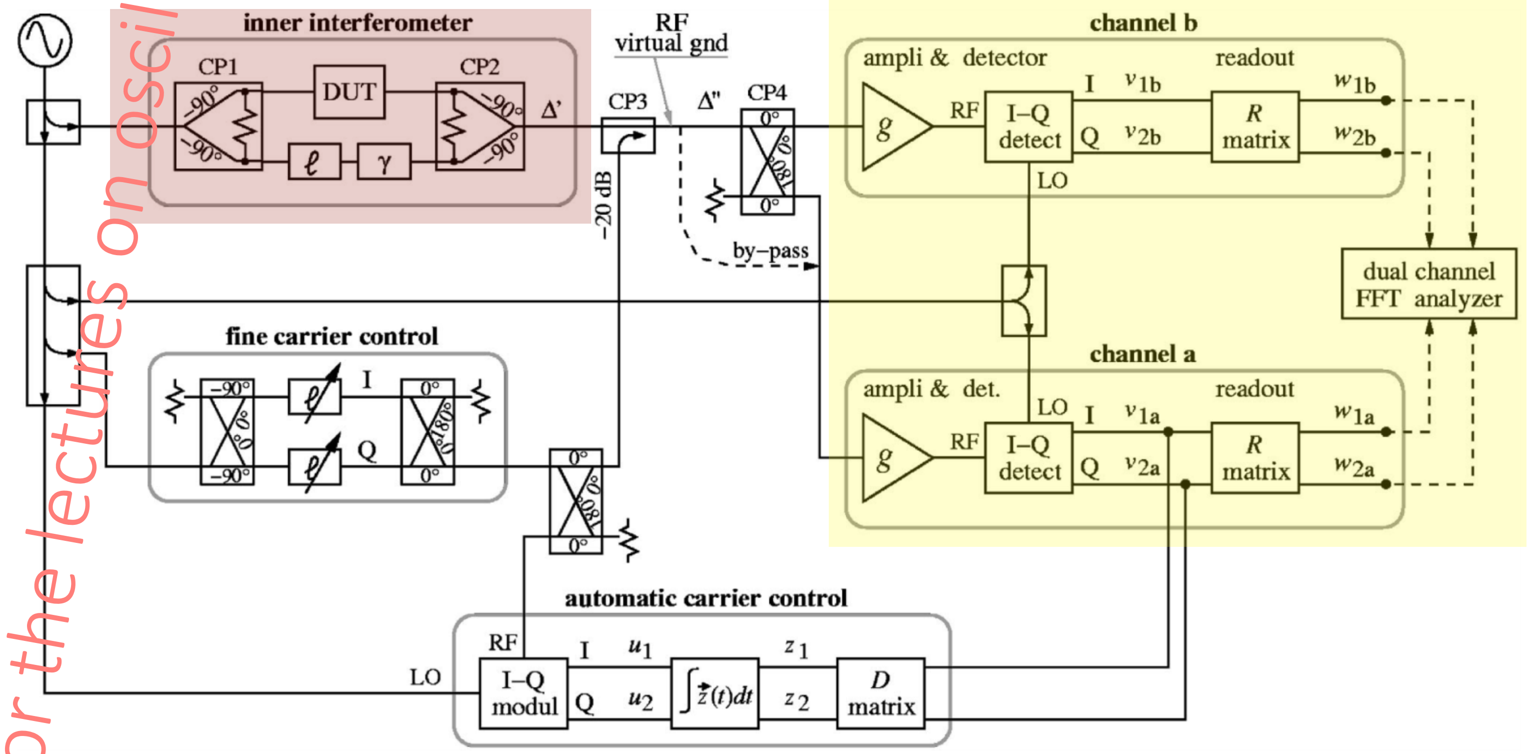
F.L. Walls & al, Proc. 30th FCS pp.269-274, 1976, Fig.8

(relatively) large correlation bandwidth
provides low noise floor in a reasonable time

F.L. Walls & al, Proc. 30th FCS pp.269-274, 1976
More popular after W. Walls, Proc. 46th FCS pp.257-261, 1992

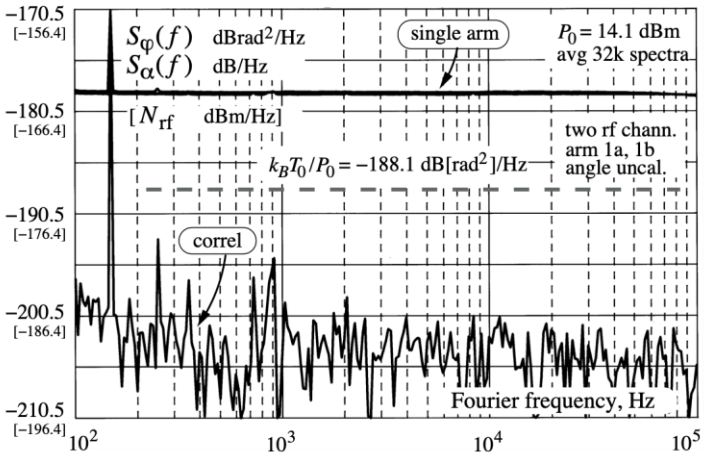
Phase Noise Measurement

E.Rubiola, V.Giordano, RSI 73(6), Jun 2002, Fig.2



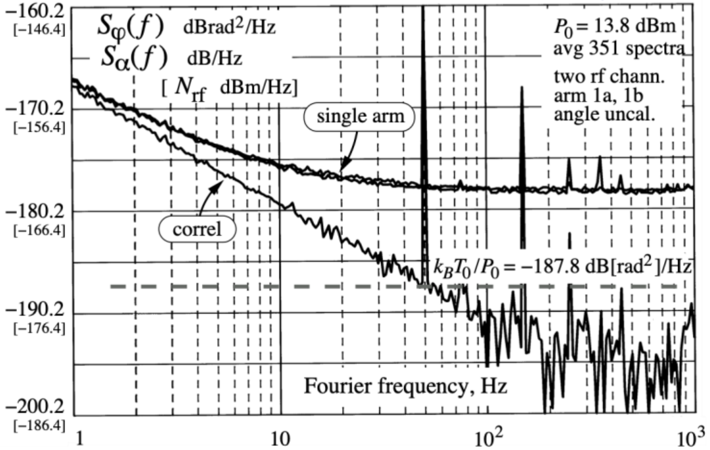
background noise

E.Rubiola, V.Giordano, RSI 73(6), Jun 2002, Fig.10



by-step attenuator

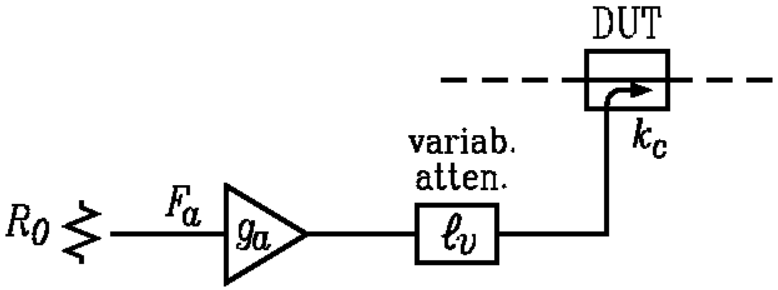
E.Rubiola, V.Giordano, RSI 73(6), Jun 2002, Fig.11



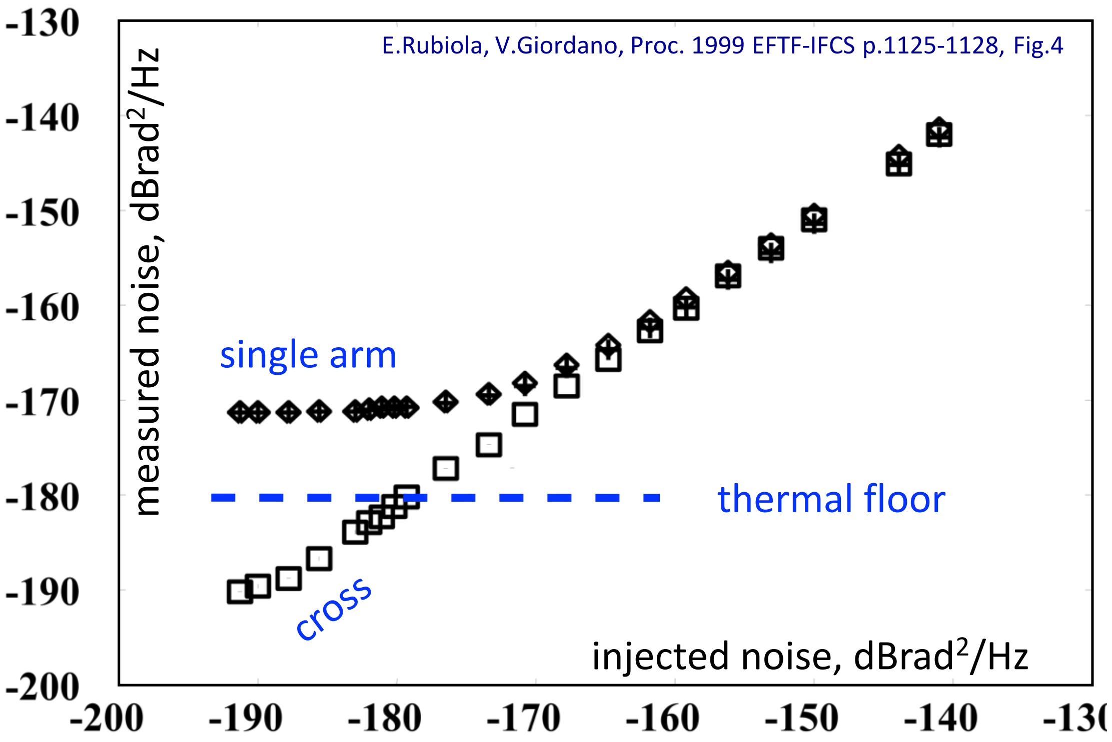
E. Rubiola, V. Giordano, Rev. Sci. Instrum. 71(8) p.3085-3091, aug 2000
E. Rubiola, V. Giordano, Rev. Sci. Instrum. 73(6) pp.2445-2457, jun 2002

Below the standard thermal floor

100 MHz prototype,
carrier power P_o = 8 dBm



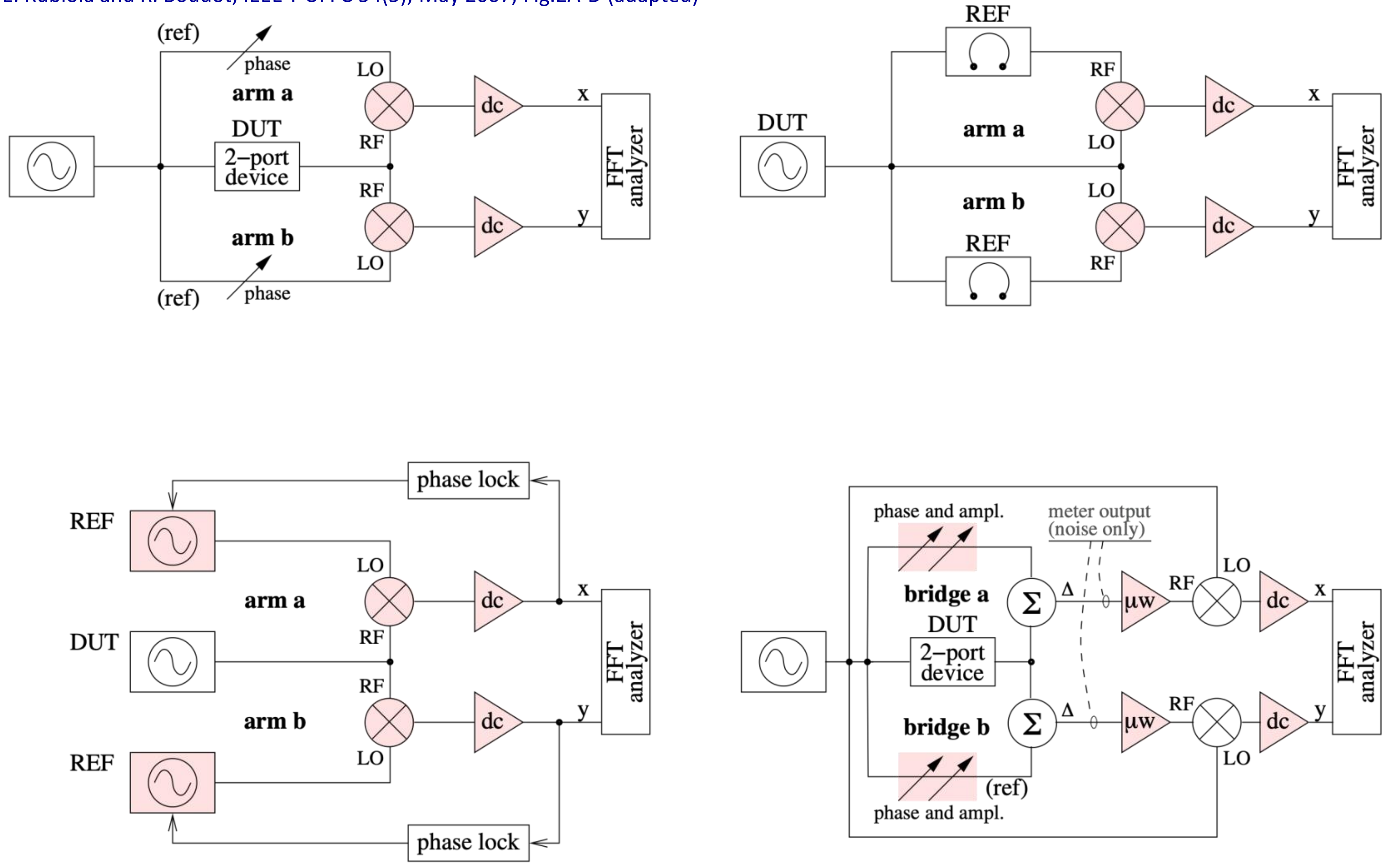
E.Rubiola, V.Giordano, Proc. 1999 EFTF-IFCS p.1125-1128, Fig.3


E.Rubiola, V.Giordano, Proc. 1999 EFTF-IFCS p.1125-1128, Fig.4

E.Rubiola, V.Giordano, Correlation-Based Noise Measurements
Below the Thermal Noise Floor, Proc. 1999 EFTF-IFCS p.1125-1128

Phase noise

E. Rubiola and R. Boudot, IEEE T UFFC 54(5), May 2007, Fig.2A-D (adapted)



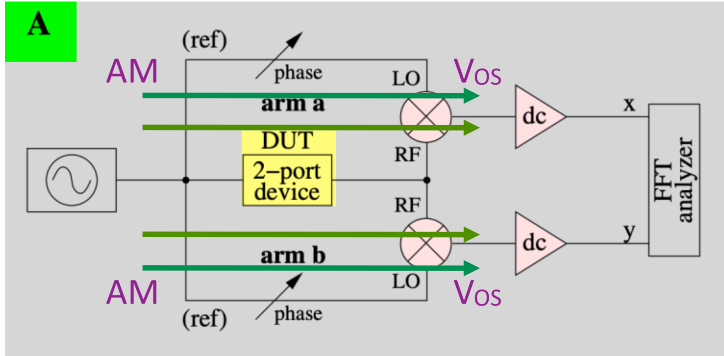
E. Rubiola and R. Boudot, The effect of AM Noise on Correlation Phase-Noise Measurements, IEEE Transact. UFFC 54(5) p.926-932, May 2007

Effect of amplitude noise

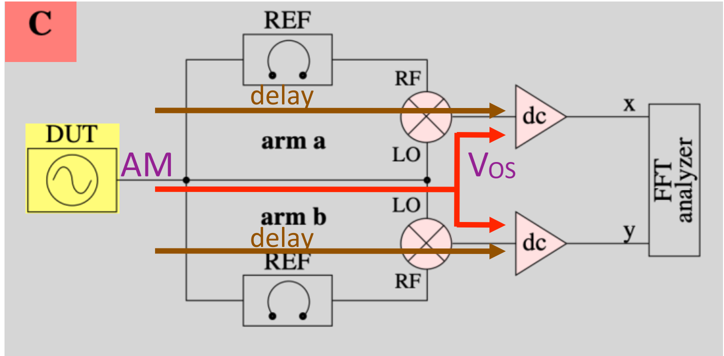
For the lectures on oscillators

E. Rubiola and R. Boudot, IEEE T UFFC 54(5), May 2007, Fig.2A-D (adapted)

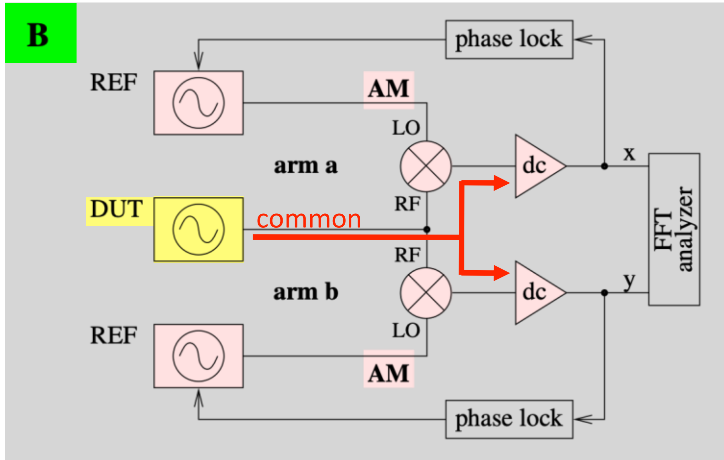
Should set both channels at the sweet point, if exists



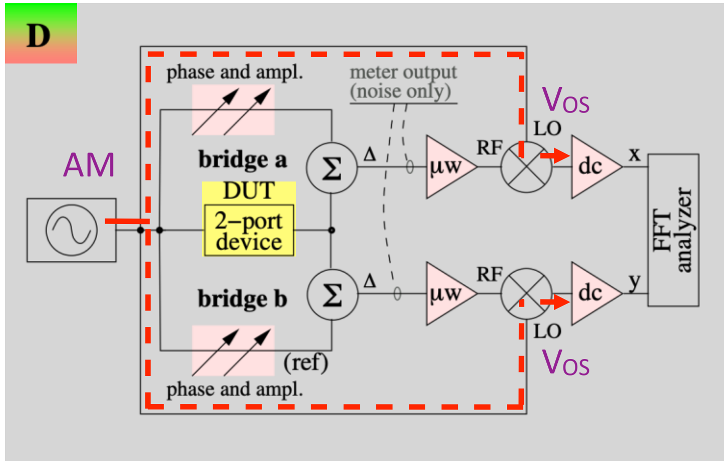
The delay de-correlates the two inputs, so there is no sweet point



pink: noise rejected by correlation and averaging



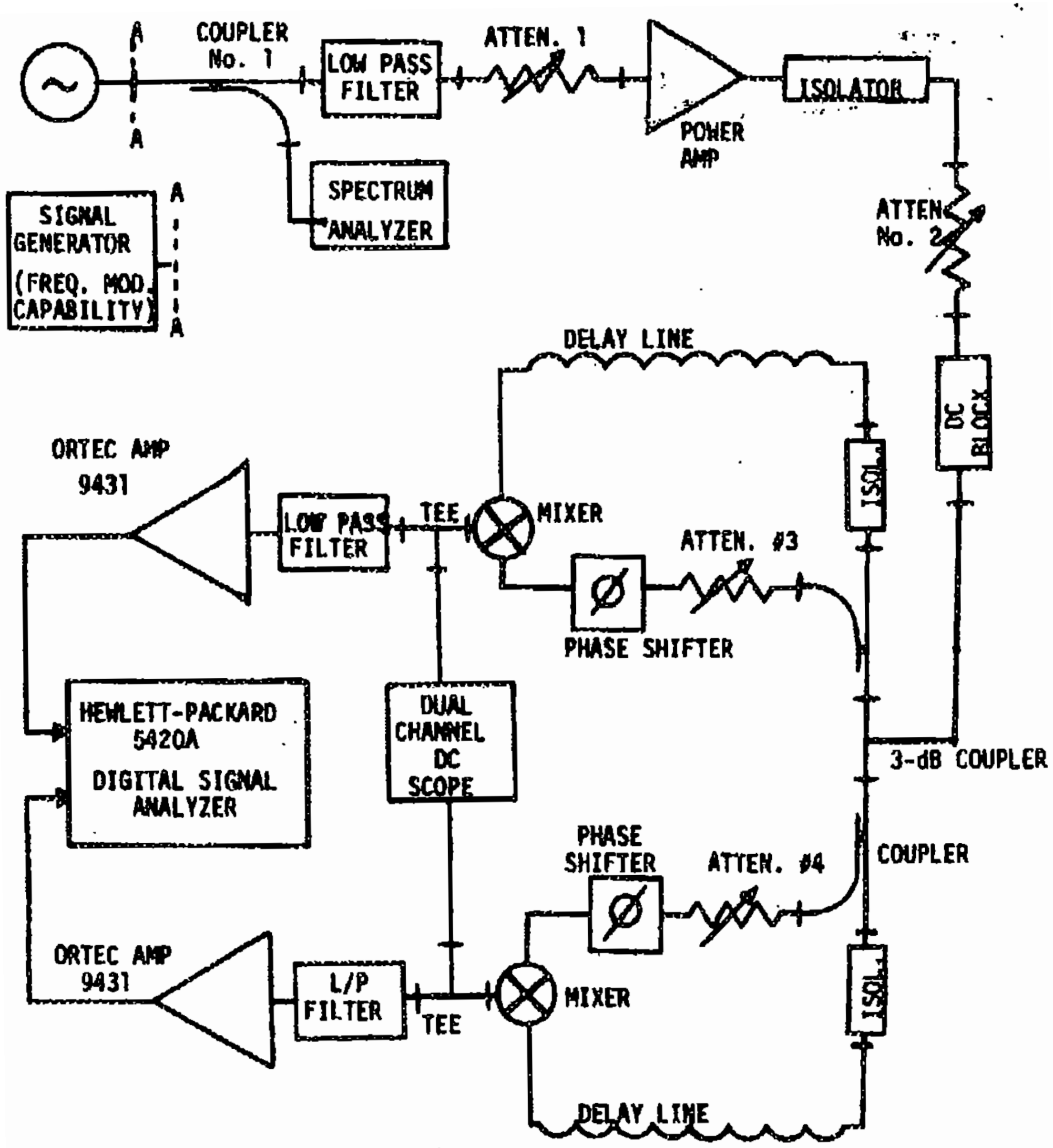
Should set both channels at the sweet point of the RF input, if exists, by offsetting the PLL or by biasing the IF



The effect of the AM noise is strongly reduced by the RF amplification

E. Rubiola and R. Boudot, The effect of AM Noise on Correlation Phase-Noise Measurements, IEEE Transact. UFFC 54(5) p.926-932, May 2007

Dual-delay-line method



A.L. Lance et al., ISA Transact. 21(4), Apr 1982, Fig.6

(arguably) Original idea by
D. Halford's NBS notebook F10 p.19-38, apr 1975

First published: A. L. Lance & al, CPEM Digest, 1978

The delay line converts the frequency noise into phase noise

The high loss of the coaxial cable limits the maximum delay

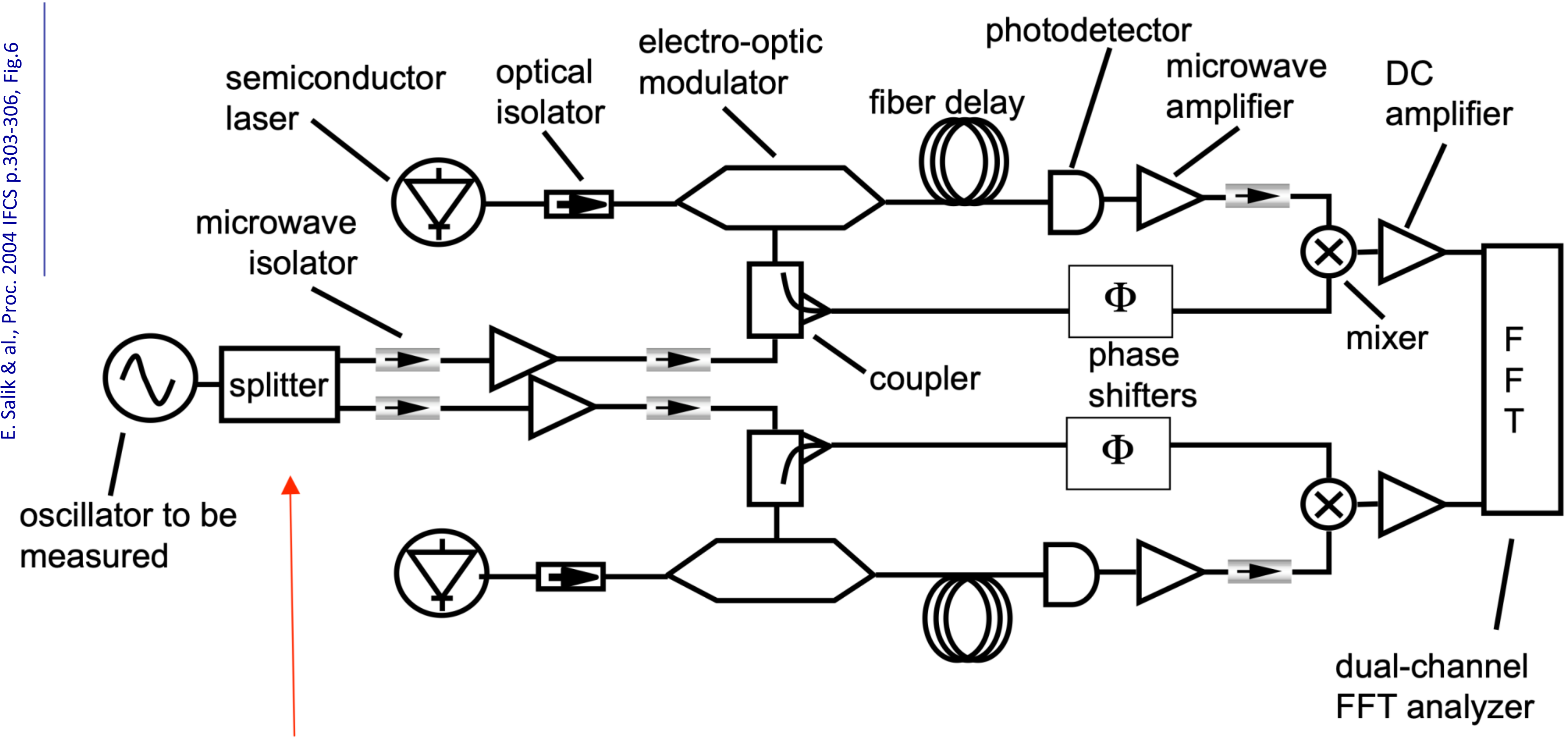
Updated version:
The optical fiber provides long delay with low attenuation
(0.2 dB/km or 0.04 dB/μs)

A.L. Lance, W.D. Seal, F. Labaar, Phase Noise Measurement Systems, ISA Transact. 21(4) p.37-84, Apr 1982

Optical dual-delay-line

Two completely separate systems measure the same oscillator under test

E. Salik & al., Proc. 2004 IFCS p.303-306, Fig.6

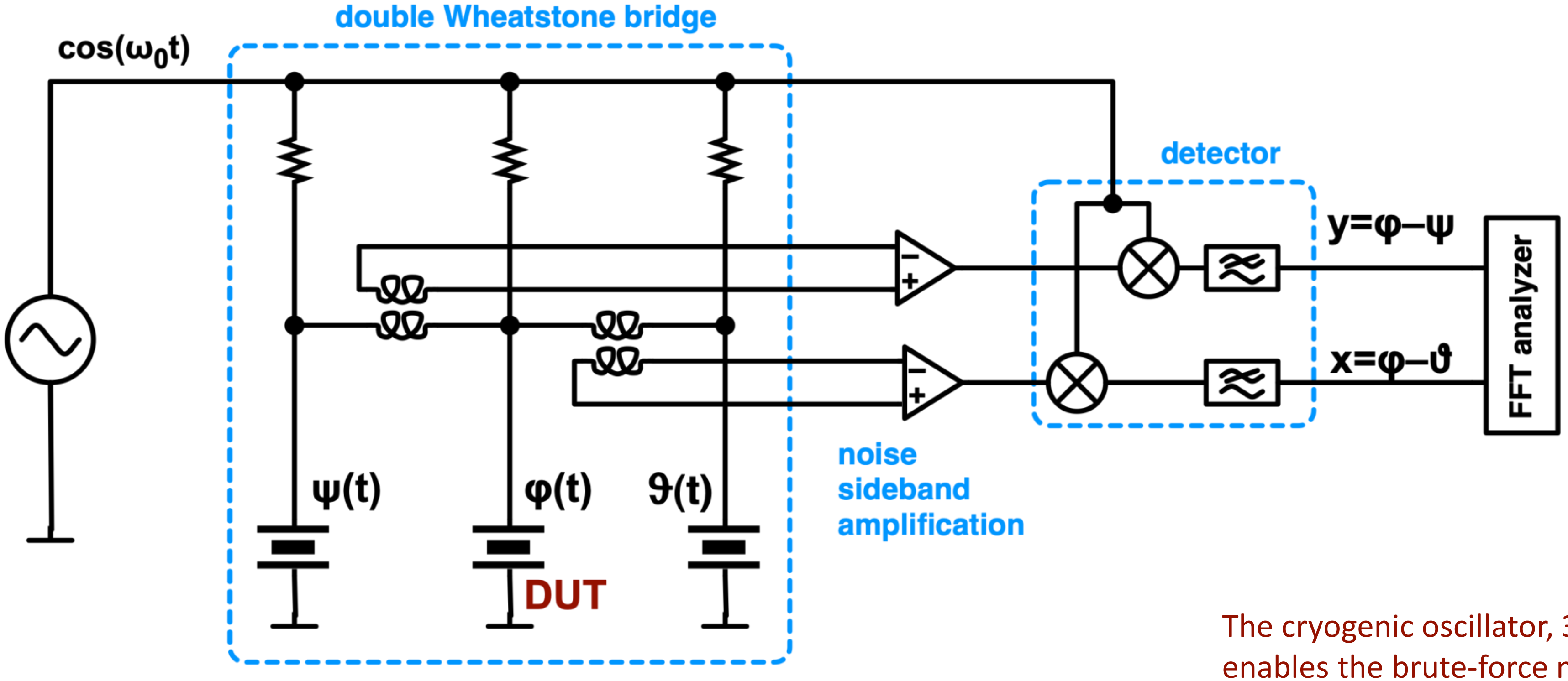


The only common part of the setup is the power splitter.

E. Salik, N. Yu, L. Maleki, E. Rubiola, Proc. IFCS, Montreal, Aug 2004 p.303-306

Volyanskiy & al., JOSAB 25(12) 2140-2150, Dec.2008. Also arXiv:0807.3494v1 [physics.optics] July 2008

Frequency stability of a resonator



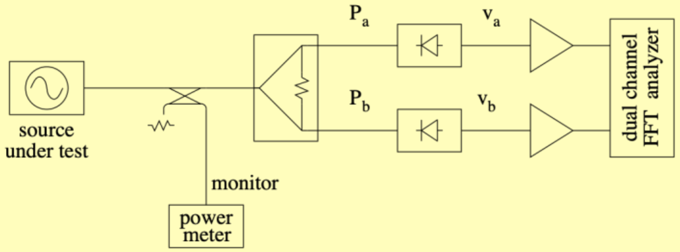
The cryogenic oscillator, 3x10^{-16} stability, enables the brute-force measurement of a single resonator

- Bridge in equilibrium
  - The amplifier cannot flicker around ω_0, which it does not know
- The fluctuation of the resonator natural frequency is estimated from phase noise
- Q matching prevents the master-oscillator noise from being taken in
- Correlation removes the noise of the instruments and the reference resonators

Amplitude noise & laser RIN

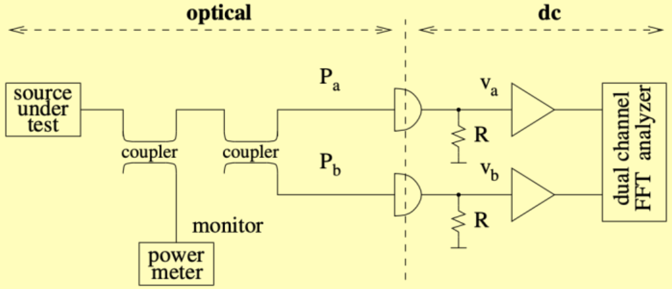
E. Rubiola, Proc. 2006
IFCS p.750-758, Fig.1

AM noise of RF/microwave sources



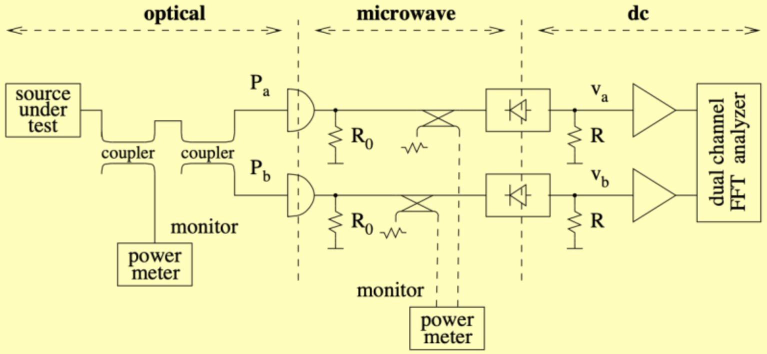
E. Rubiola, Proc. 2006
IFCS p.750-758, Fig.7

Laser RIN

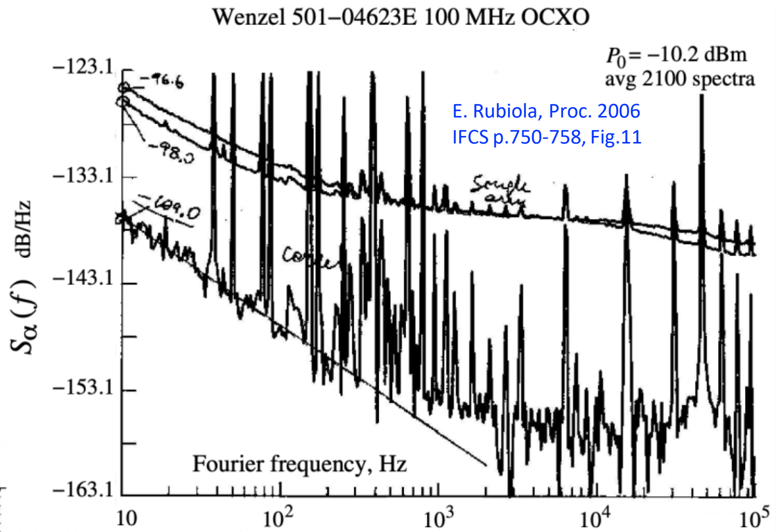


E. Rubiola, Proc. 2006
IFCS p.750-758, Fig.8

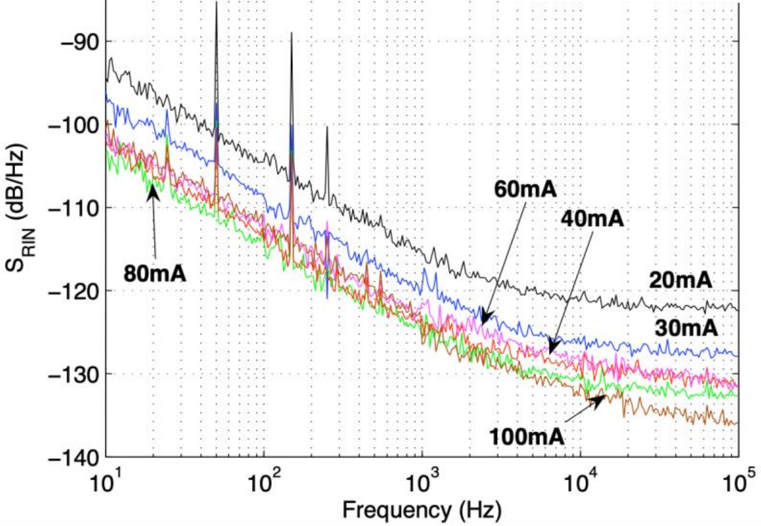
AM noise of photonic RF/microwave sources



- Cannot measure the background removing the DUT
- Correlation enables to validate the instrument



E. Rubiola, Proc. 2006
IFCS p.750-758, Fig.11

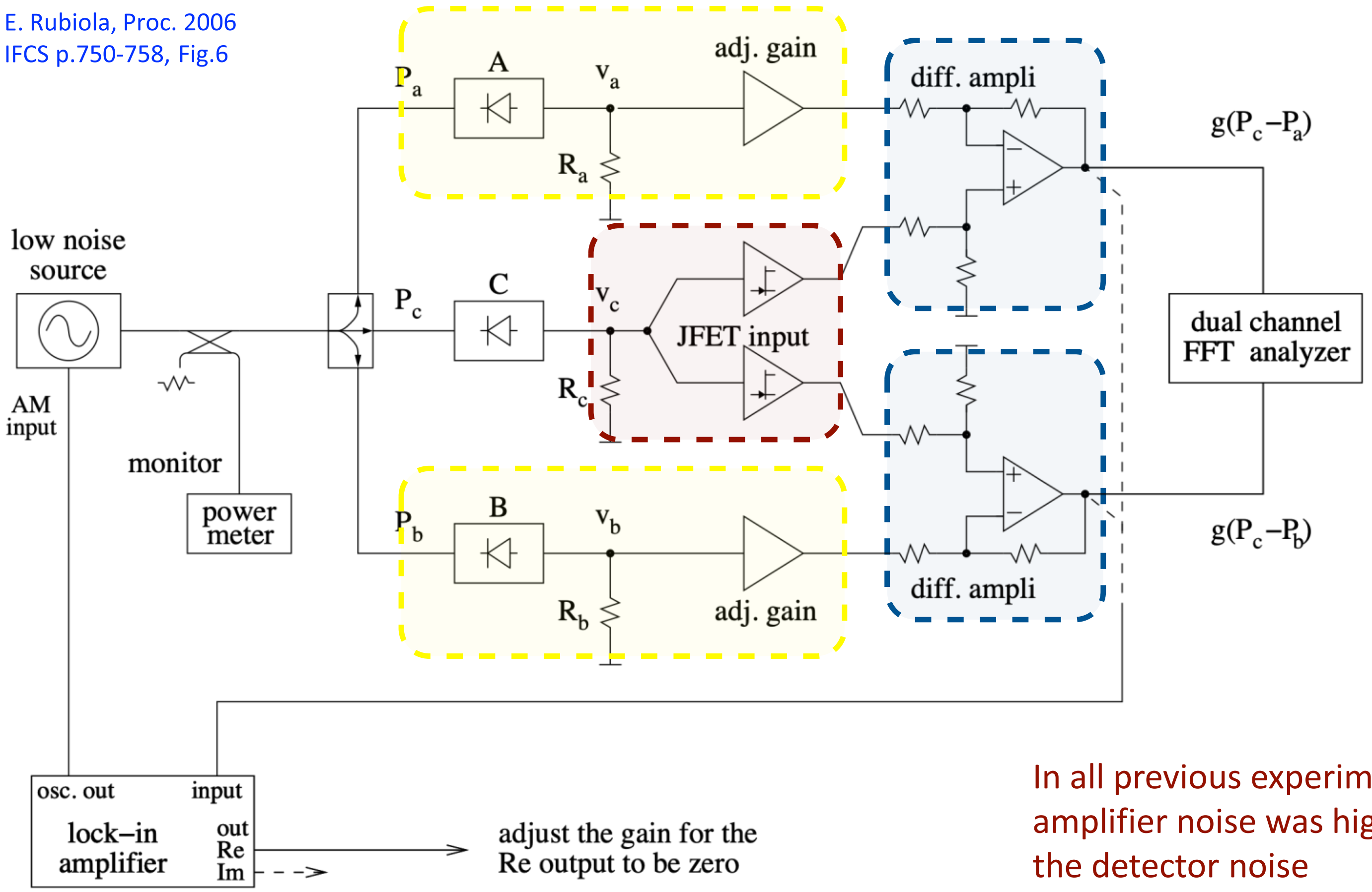


Courtesy of Kirill Volyanskiy, 2009

E. Rubiola, The measurement of AM noise, Proc. IFCS p.750-758, June 2006.
Also arXiv:physics/0512082v1 [physics.ins-det], Dec 2005

Detector noise

E. Rubiola, Proc. 2006
IFCS p.750-758, Fig.6



Basic ideas

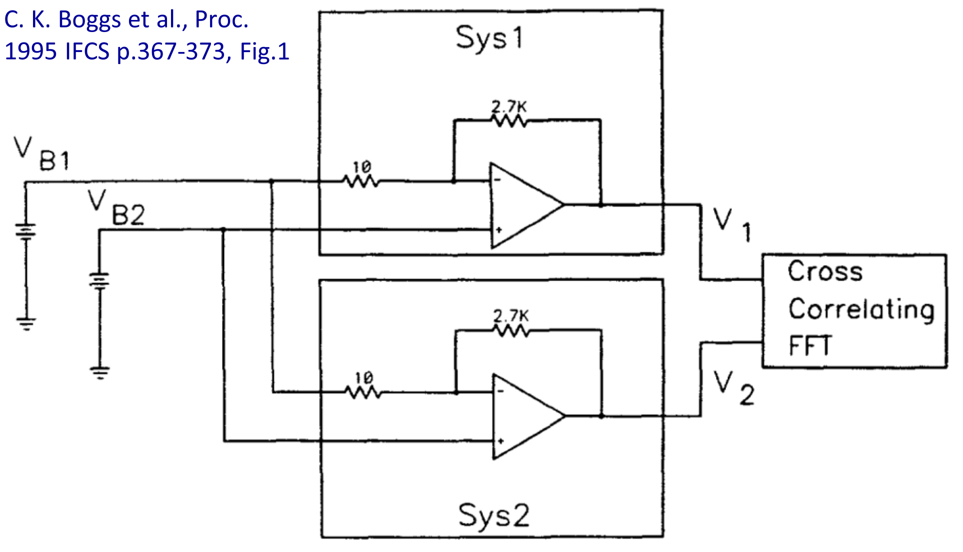
- Remove the noise of the source by balancing C–A and C–B
  - Use a lock-in amplifier to get a sharp null measurement
- Channels A and B are independent –> noise is averaged out
- Two separate JFET amplifiers are needed in the C channel
  - JFETs have virtually no bias-current noise
- Only the noise of the detector C remains

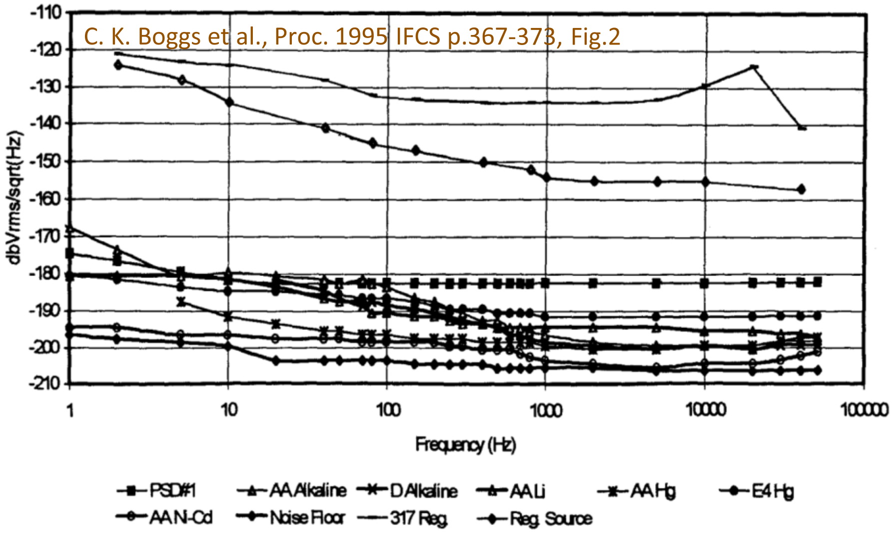
In all previous experiments, the amplifier noise was higher than the detector noise

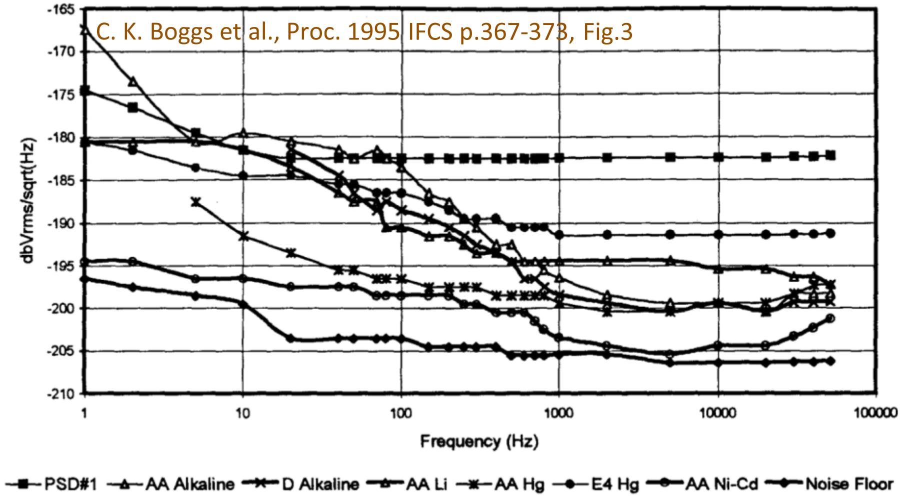
E. Rubiola, The measurement of AM noise, Proc. IFCS p.750-758, June 2006. Also arXiv:physics/0512082v1 [physics.ins-det], Dec 2005

S. Grop, E. Rubiola, Flicker Noise of Microwave Power Detectors, Proc. IFCS p.40-43, April 2009

Noise in chemical batteries


C. K. Boggs et al., Proc. 1995 IFCS p.367-373, Fig.1


C. K. Boggs et al., Proc. 1995 IFCS p.367-373, Fig.2


C. K. Boggs et al., Proc. 1995 IFCS p.367-373, Fig.3

- Do not waste DAC bits for a constant DC, $V = V_{B2}–V_{B1}$ has (almost) zero mean
- Two separate amplifiers measure the same quantity V
- Correlation rejects the amplifier nose, and the FFT noise as well

C. K. Boggs, A. D. Doak, F. L. Walls, Measurement of Voltage Noise in Chemical Batteries, Proc. IFCS p.367-373, 1995

Noise in semiconductors

M. Sampietro & al., RSI 70(5) p.2520-2525, May 1999, Fig.2

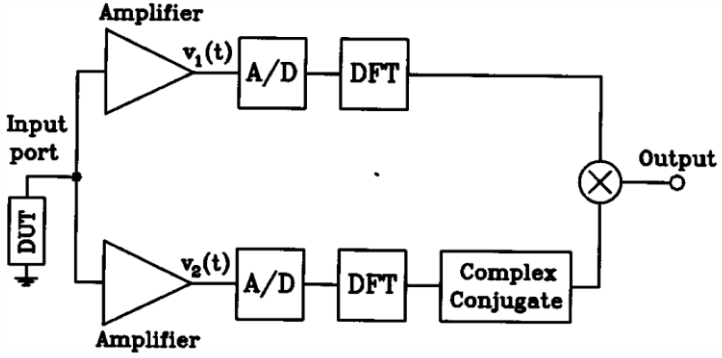


FIG. 2. Schematics of the building blocks of our correlation spectrum analyzer performing the suppression of the uncorrelated input noises by a digital processing of sampled data.

M. Sampietro & al., RSI 70(5) p.2520-2525, May 1999, Fig.3

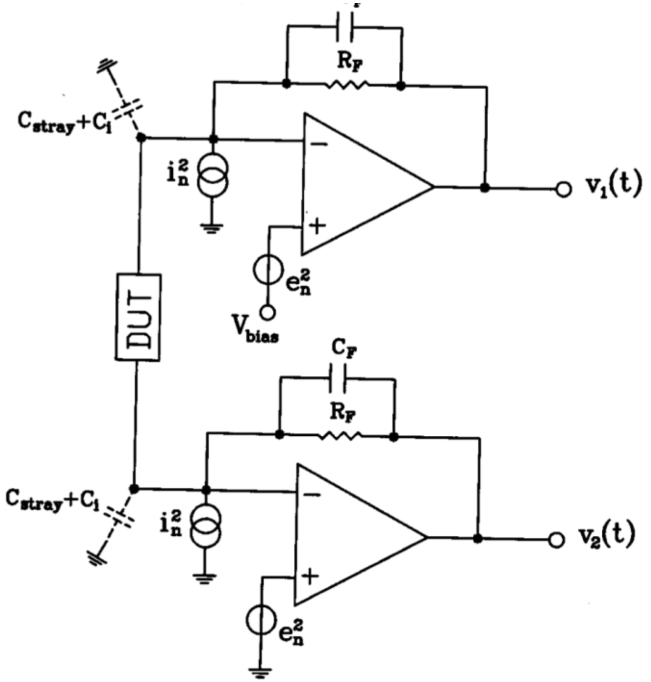


FIG. 3. Schematics of the active test fixture for current noise measurements.

M. Sampietro & al., RSI 70(5) p.2520-2525, May 1999, Fig.9

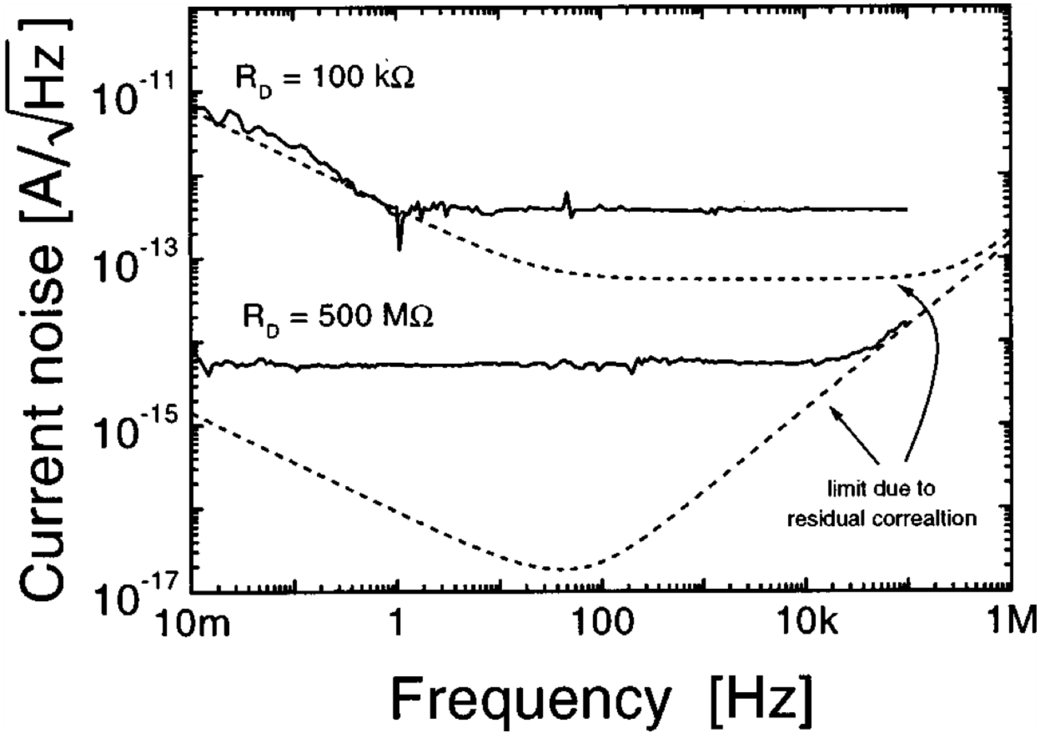
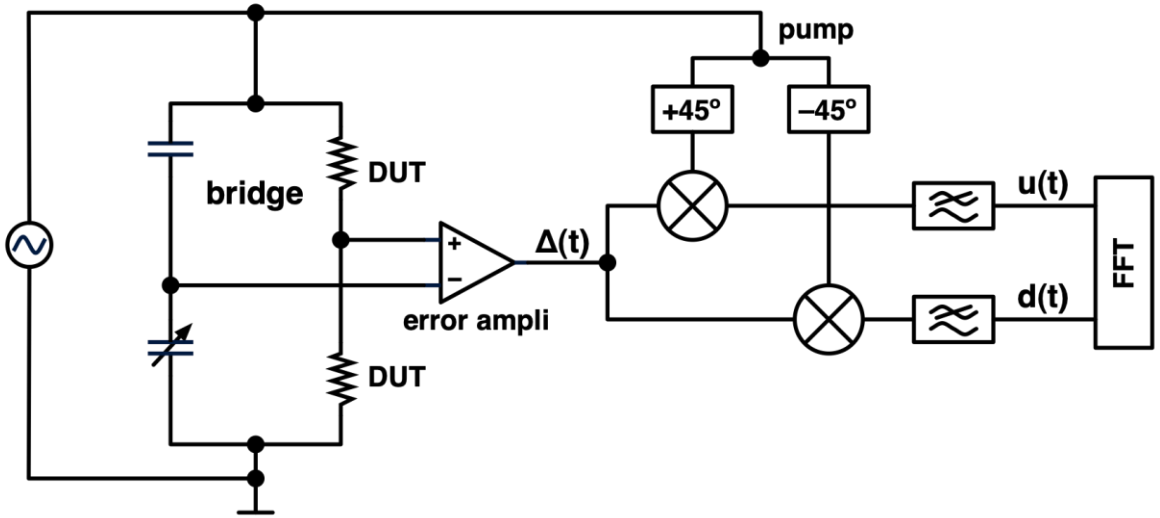


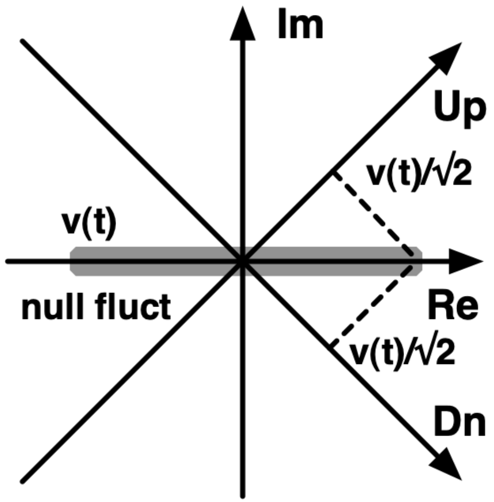
FIG. 9. Experimental frequency spectrum of the current noise from DUT resistances of 100 kΩ and 500 MΩ (continuous line) compared with the limits (dashed line) given by the instrument and set by residual correlated noise components.

Sampietro M, Fasoli L, Ferrari G - Spectrum analyzer with noise reduction by cross-correlation technique on two channels - RSI 70(5) p.2520-2525, May 1999

Electro-migration in thin films



- Random noise: X' and X'' (real and imag part) of a signal are statistically independent
- The detection on two orthogonal axes eliminates the amplifier noise.
  This work with a single amplifier!
- The DUT noise is detected



$$S_{ud}(f) = \frac{1}{2}\left[S_{\alpha}(f) - S_{\varphi}(f)\right]$$

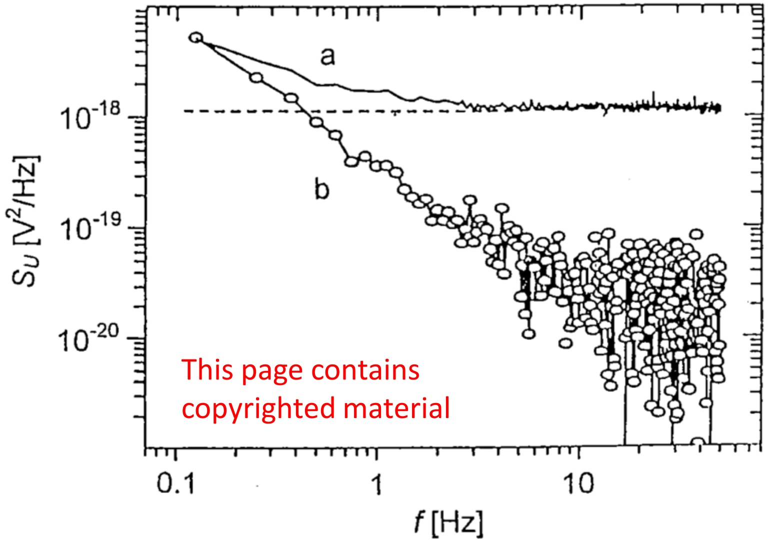


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Fig. 1 1/f noise of an AlSi$_{0.01}$Cu$_{0.002}$ thin film measured at room temperature (a) without and (b) with the phase-sensitive ac correlation technique. The Johnson noise level is indicated by the dashed line.

A. Seeger, H. Stoll, Proc. 1999 ICNF p.162-167, Fig.1

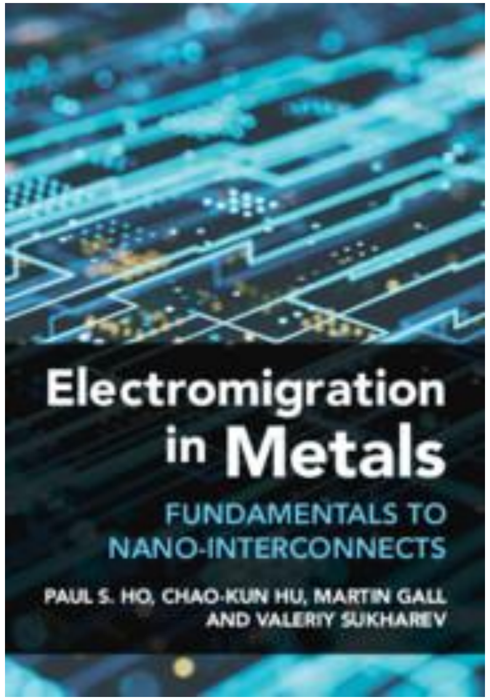
A. Seeger, H. Stoll, 1/f noise and defects in thin metal films, Proc. ICNF p.162-167, Hong Kong 23-26 aug 1999

RF/microwave version: E. Rubiola, V. Giordano, H. Stoll, IEEE Transact. IM 52(1) pp.182-188, feb 2003

Electromigration in metals is still a hot topic

Paul S. Ho, Chao-Kun Hu, Martin Gall, Valeriy Sukharev, Siemens, *Electromigration in Metals*, Cambridge, May 2022

ISBN: 9781107032385

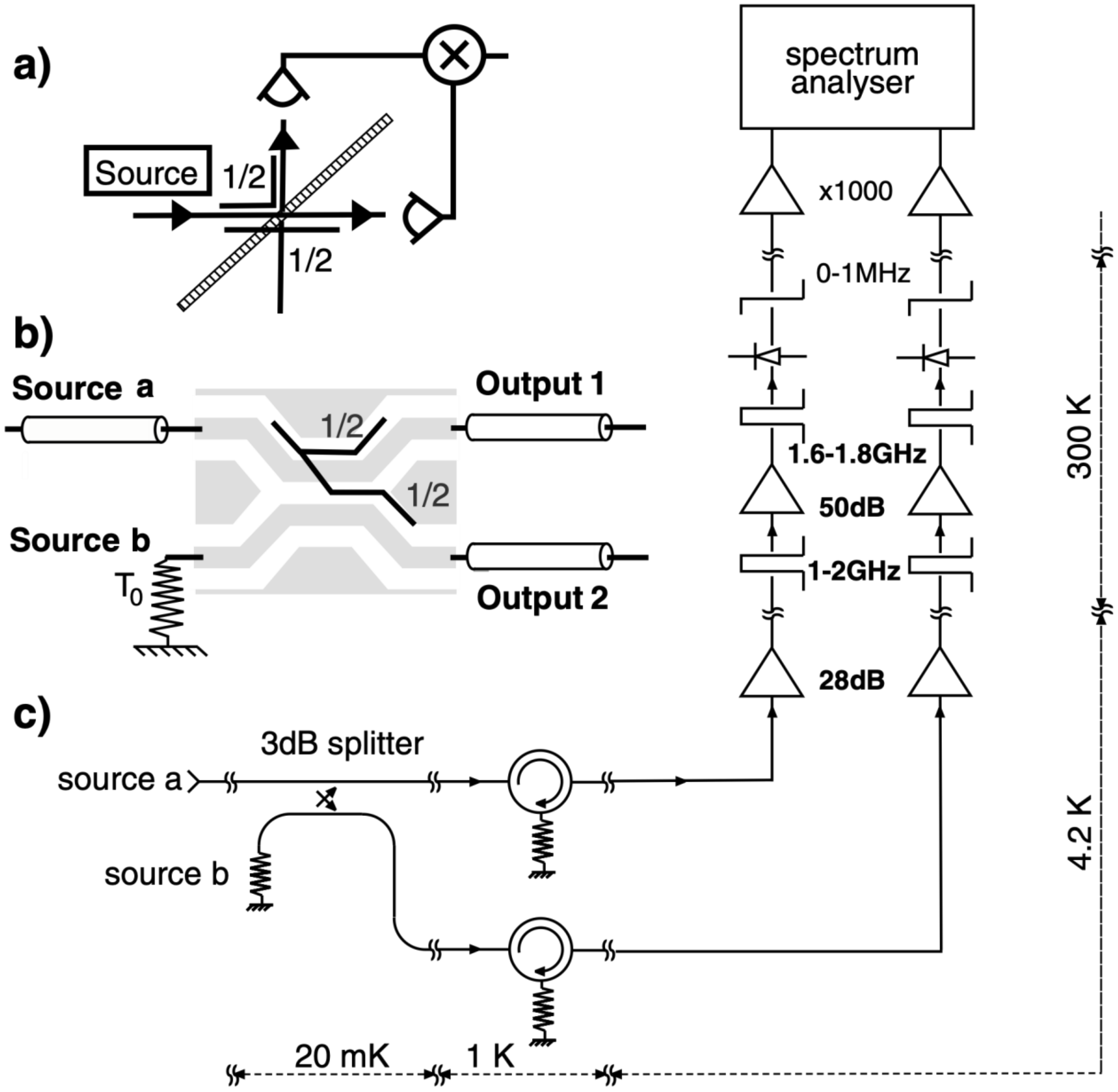


Hanbury Brown – Twiss Effect

Anti-correlation shows up in single-photon regime

Also observed in microwaves
Gabelli…Glattli, PRL 93(5) 056801, Jul 2004

20 mK and 1.7 GHz
$kT = 2.7\times10^{-25}$ J
$h\nu = 1.12\times10^{-24}$ J
$kT/h\nu = -6.1$ dB



J.Gabelli et al, PRL 93(5) 056801, 27 July 2004, Fig.1

Featured reading (optics)
Hanbury Brown R, Twiss RQ - Correlation Between Photons in Two Coherent Beams of Light - Nature 4497 p.27-29, 7 January 1956

Featured reading (microwave port)
Gabelli J, Reydellet LH, Feve G, Berroir JM, Placais B, Roche P, Glattli DC, Hanbury-Brown Twiss Correlation to Probe the Population Statistics of GHz Photons Emitted by Conductors, PRL 93(5) 056801, 27 July 2004

Lecture 4 ends here

Lecture 5

Scientific Instruments & Oscillators

Lectures for PhD Students and Young Scientists

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Contents

- Spectrum analyzer
- Lock-in amplifiers and boxcar average
- Frequency-to-digital and time-to-digital converters

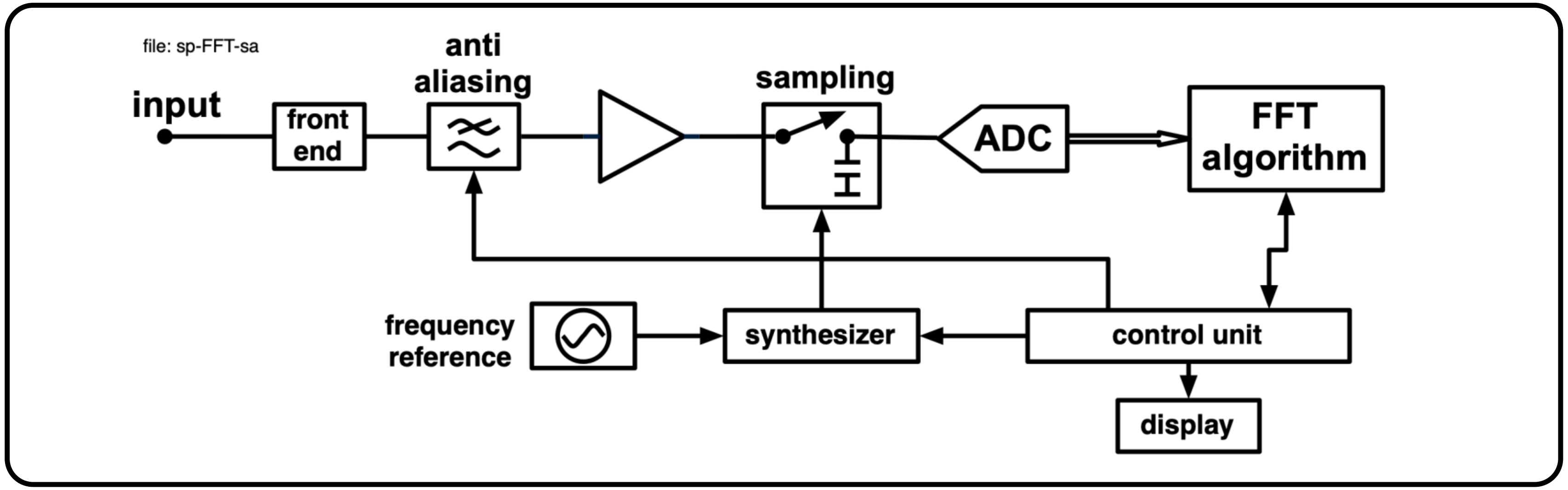
ORCID 0000-0002-5364-1835

home page http://rubiola.org

Spectrum Analyzers

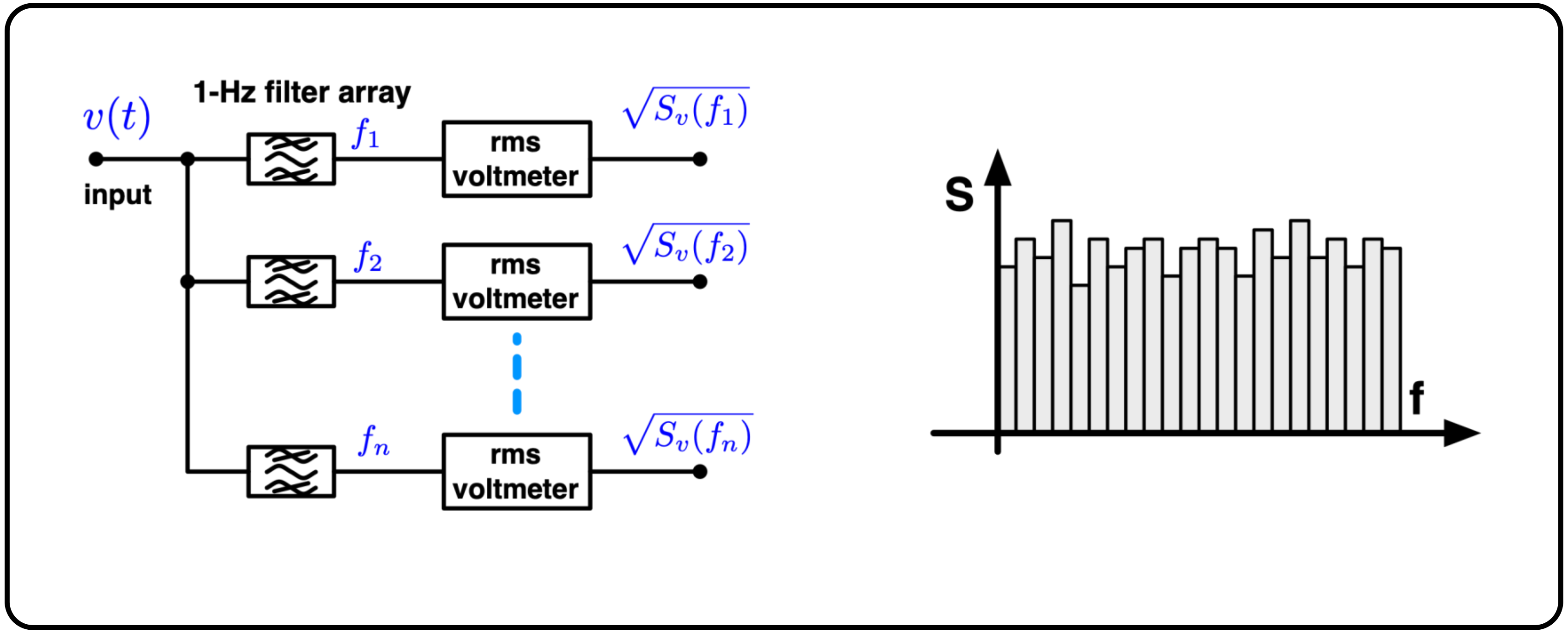
Excerpt from 03 Power Spectra

FFT spectrum analyzer



- Direct digitization of the input signal
- Fully digital process
- Practical limit $f \leq 0.4\ f_s$
- Tough tradeoff between resolution and max frequency

Parallel spectrum analyzer



Rice representation

Integration over a finite time

$$x(t) = \sum_{n=0}^{\infty} a_n(t)\cos(n\omega_0 t) - b_n(t)\sin(n\omega_0 t)$$

$$S_x(n\omega_0) = \left[a_n^2 + b_n^2\right]/\omega_0$$

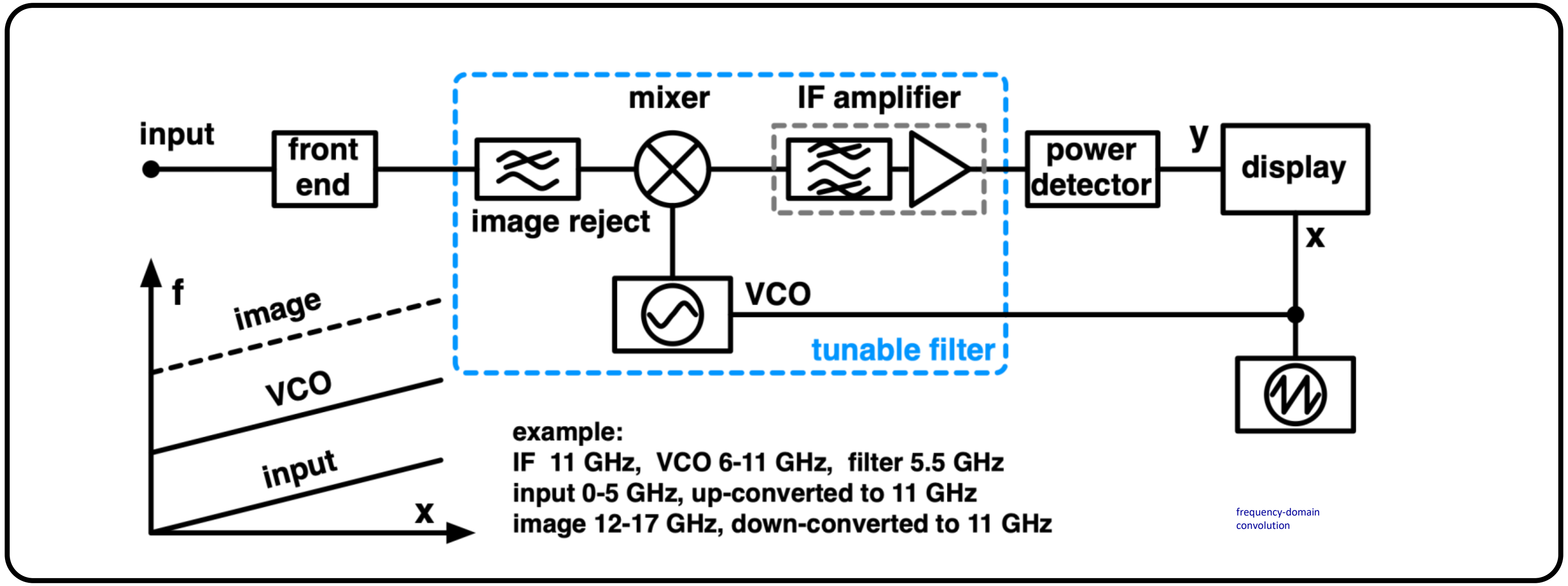
ω_0 is the analysis bandwidth

Vibrating-reed frequency meter

Julo, 24 mar 2006
(Wikimedia Commons)

Mass & Spring —> resonator

Scanning spectrum analyzer

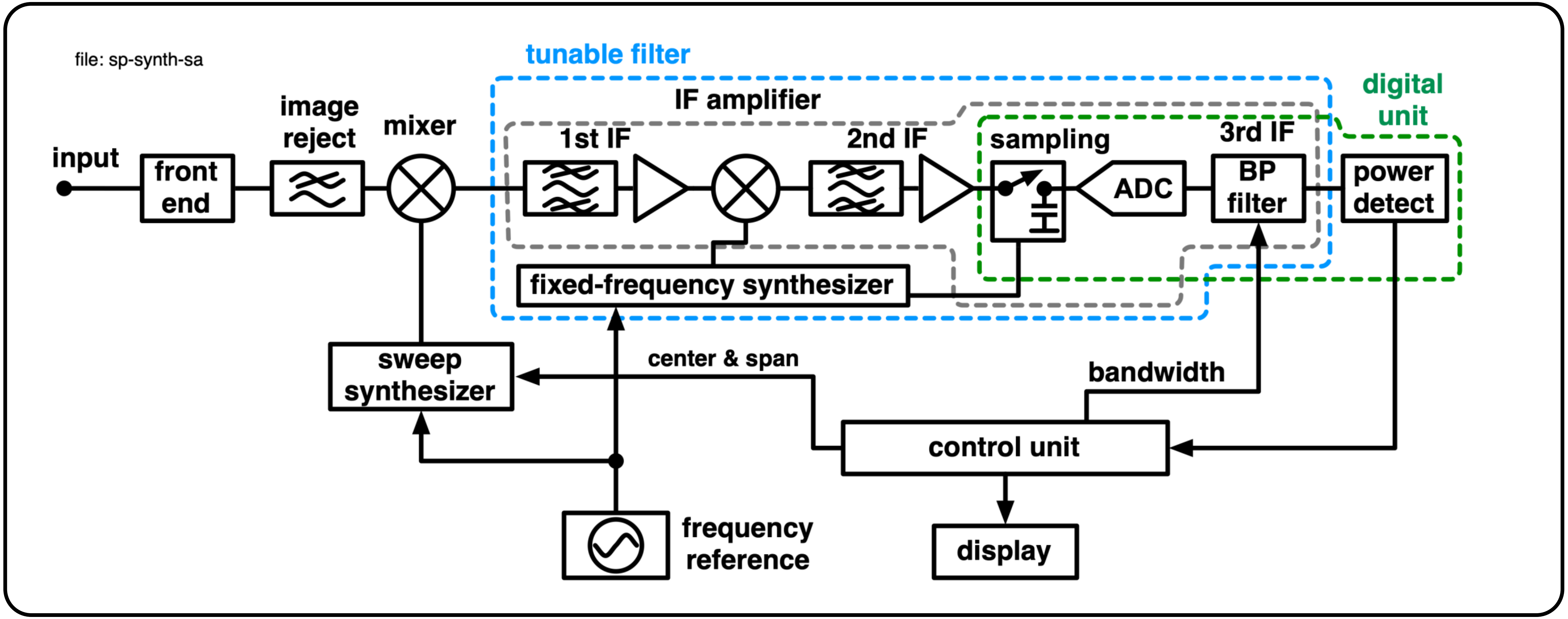


example:
IF 11 GHz, VCO 6-11 GHz, filter 5.5 GHz
input 0-5 GHz, up-converted to 11 GHz
image 12-17 GHz, down-converted to 11 GHz

frequency-domain convolution

- RF/microwaves
  - The one and only option until the late 1990s
  - Progressively replaced with the hybrid analyzer
- Optics
  - Cannot use IF
  - Analog VCO — tunable laser

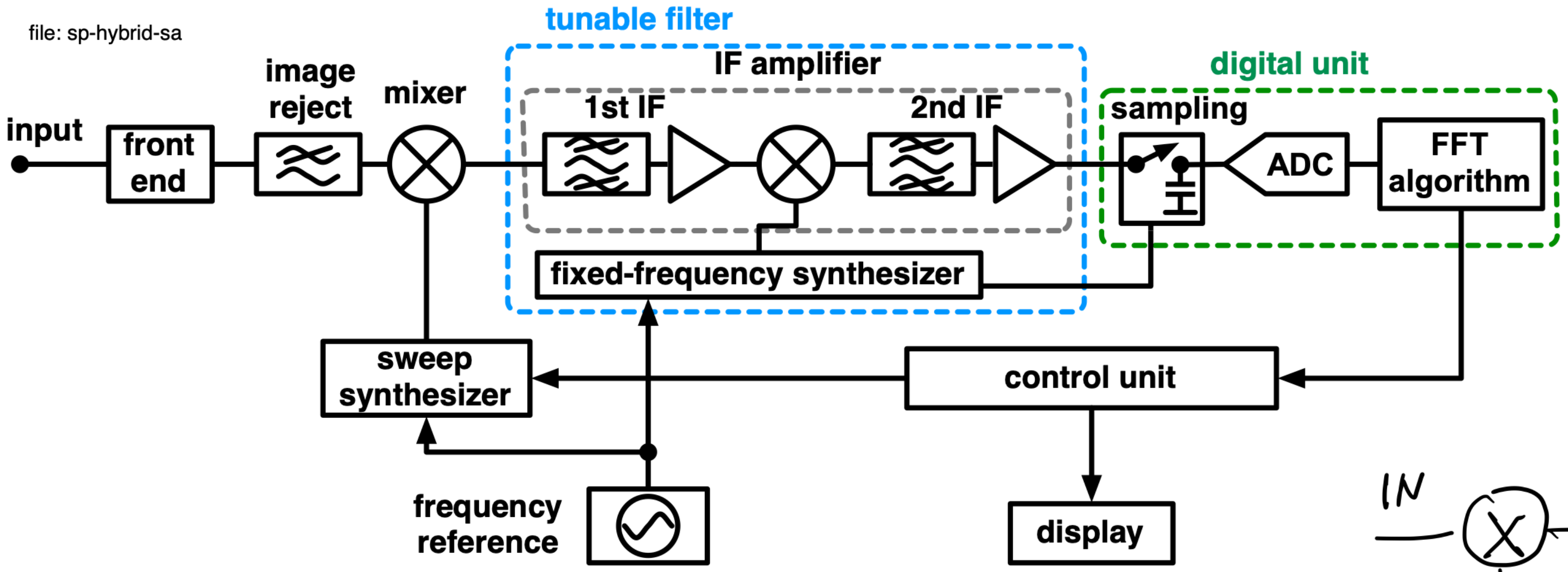
Synthesized spectrum analyzer



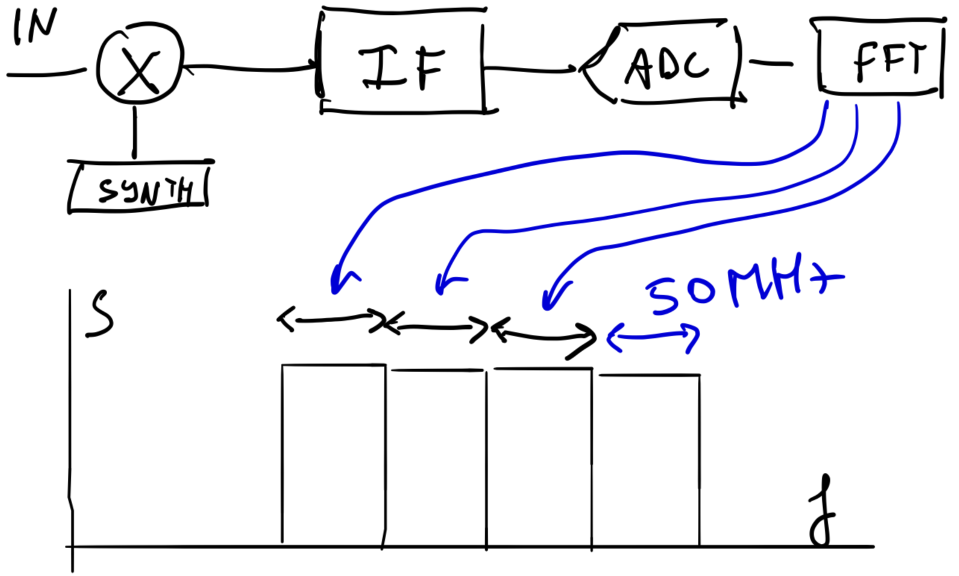
- The VCO is replaced with a synthesizer
- Otherwise similar to the scanning SA

Hybrid FFT spectrum analyzer

file: sp-hybrid-sa



- The synthesizer sweeps in wide steps
- FFT analysis in each step provides the resolution



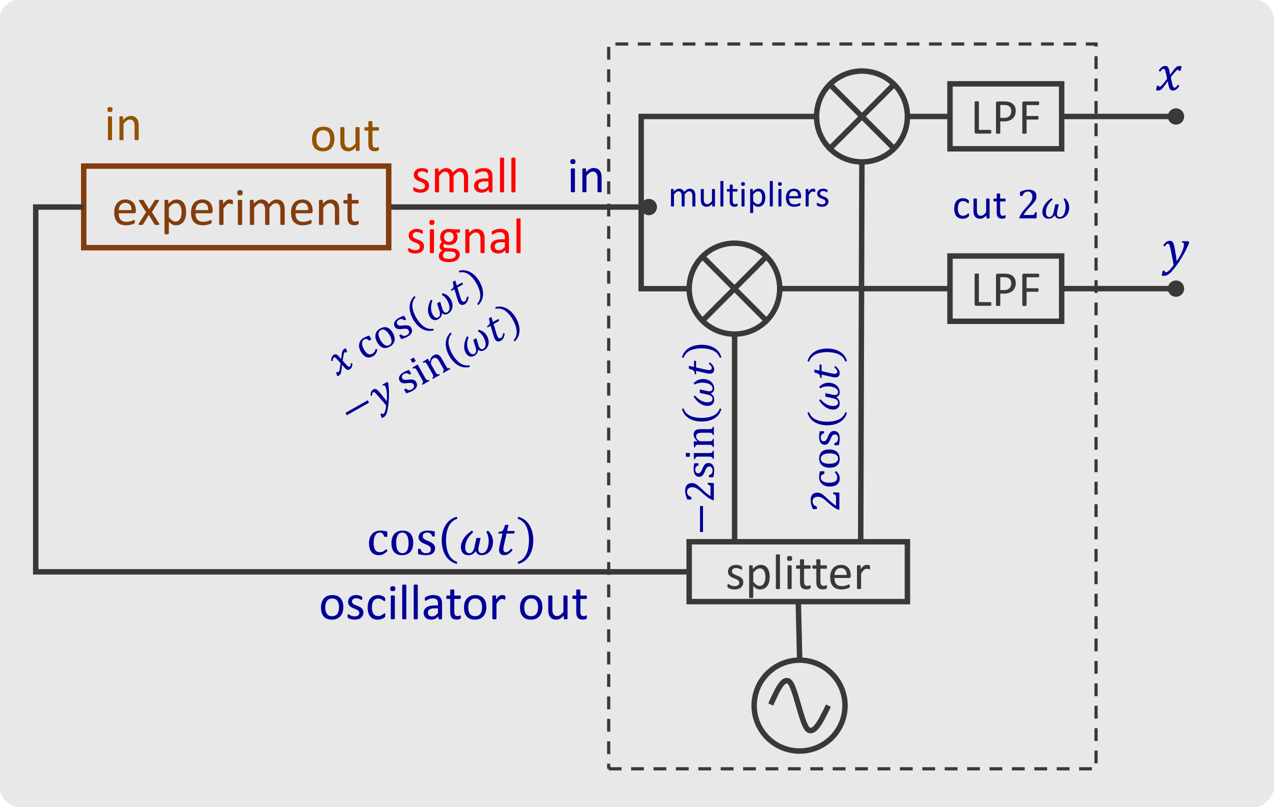
FFTs taken sequentially and joined

Lock–in Amplifier

Lock-in Amplifier – main ideas

1. Very small signal
    1. Can be detected if you have the reference
2. AC measurement:
    - Get out of the DC, drift and flicker region
3. Differential measurement
    - Oscillator is common mode
    - Fluctuations rejected
4. Transposed filter solves
    - Narrow bandwidth
    - Shape
    - Stability of center frequency and bandwidth

Next year: Explain what is the signal, and in/out. Fourier components

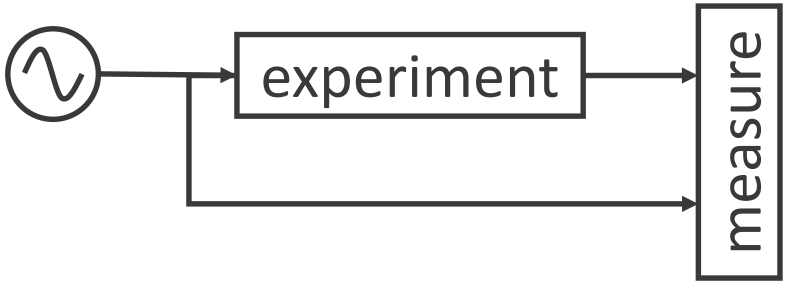


Narrowband $x(t)$ and $y(t)$

$$\{[x(t)\cos(\omega t) - y(t)\sin(\omega t)] \times 2\cos(\omega_t)\} * \text{LPF} = x(t)$$

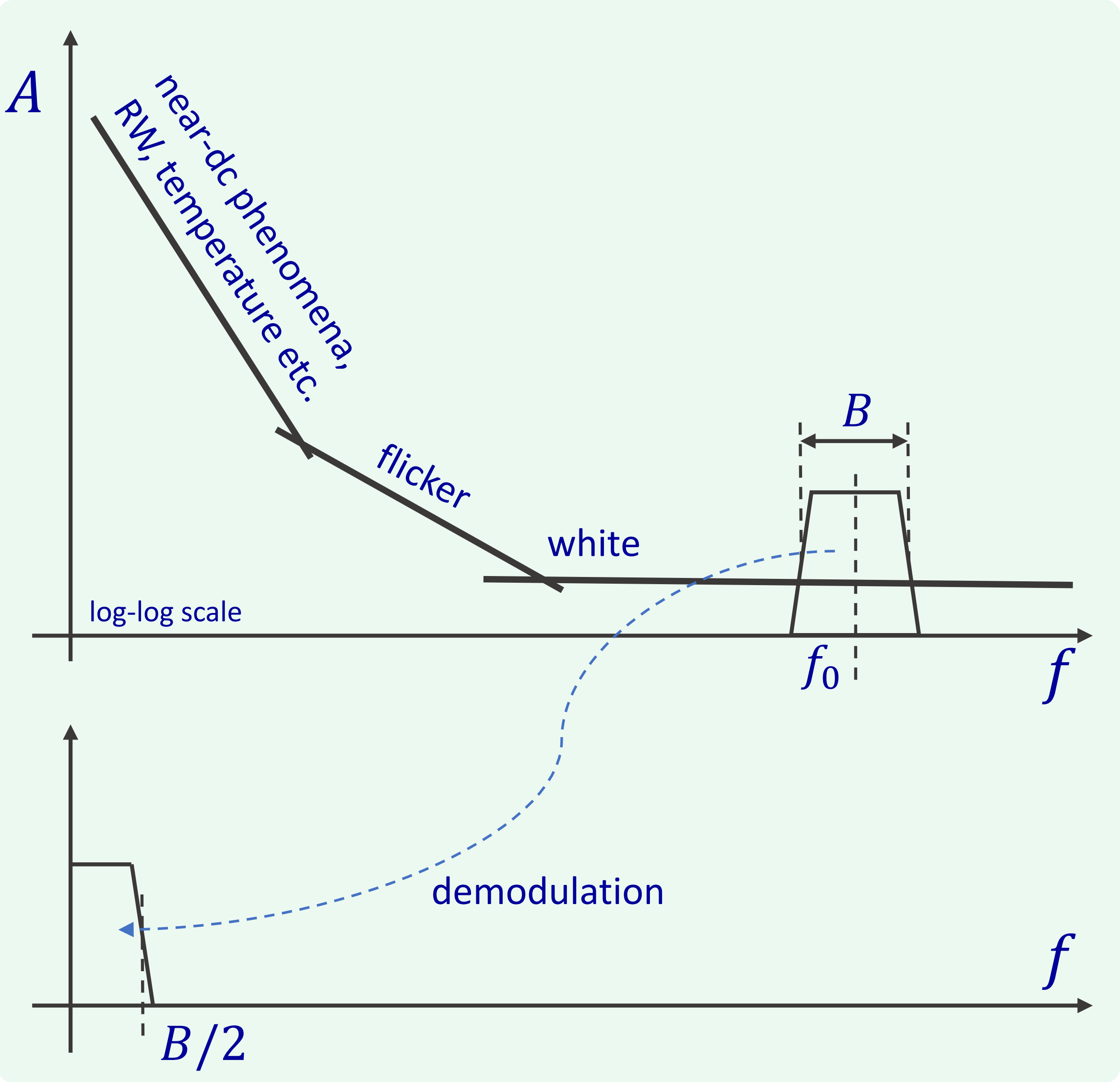
$$\{[x(t)\cos(\omega t) - y(t)\sin(\omega t)] \times [-2\sin(\omega_t)]\} * \text{LPF} = y(t)$$

Synchronous detection

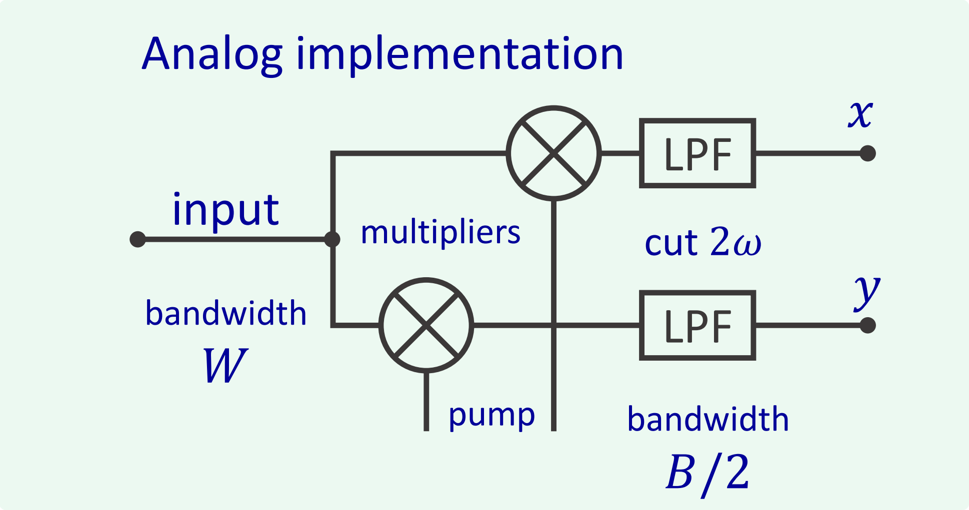


Physical property

- Transparence
- Attenuation
- Resonance
- Molecular absorption
- Capacitance
- Resistance
- etc.

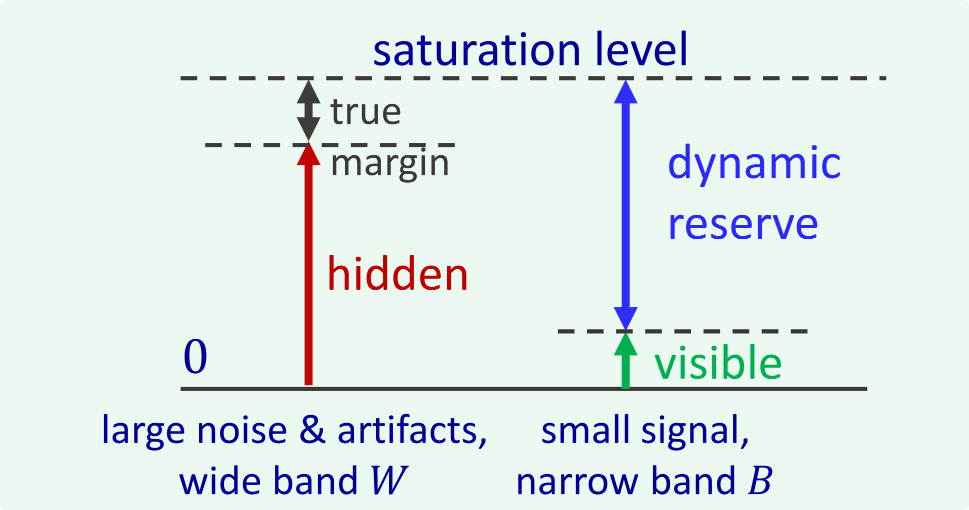


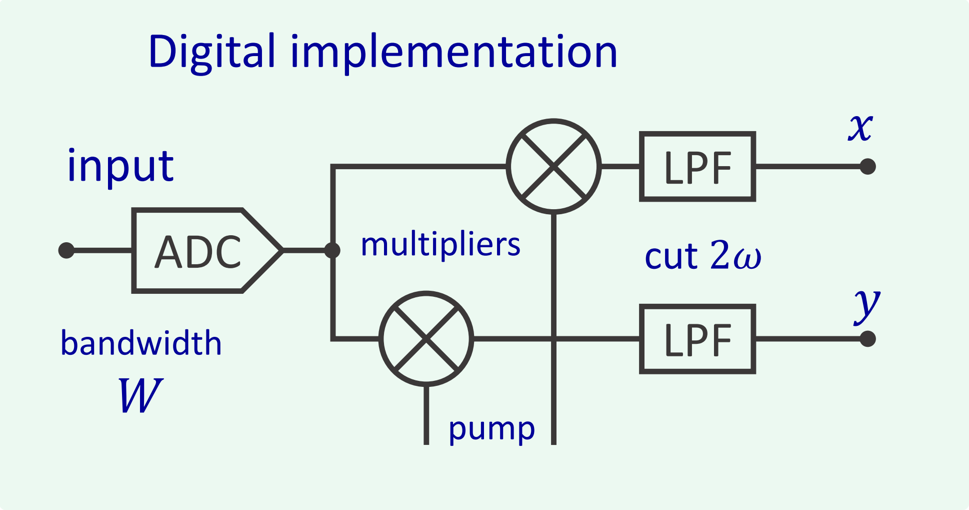
Dynamic Reserve



problem

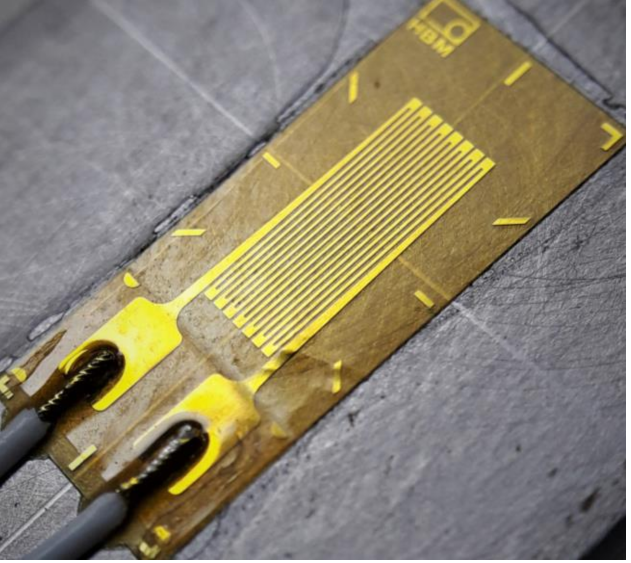
$W \gg B$



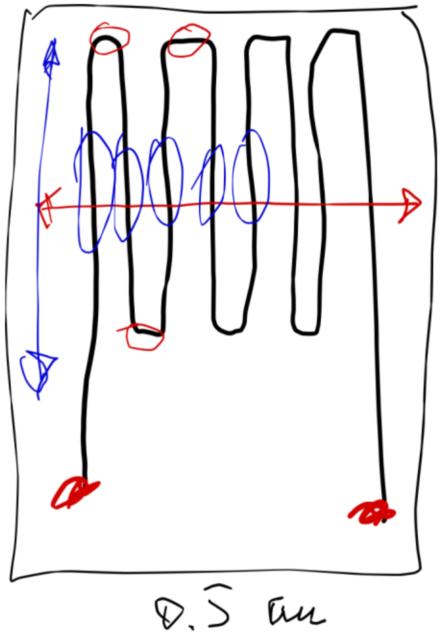


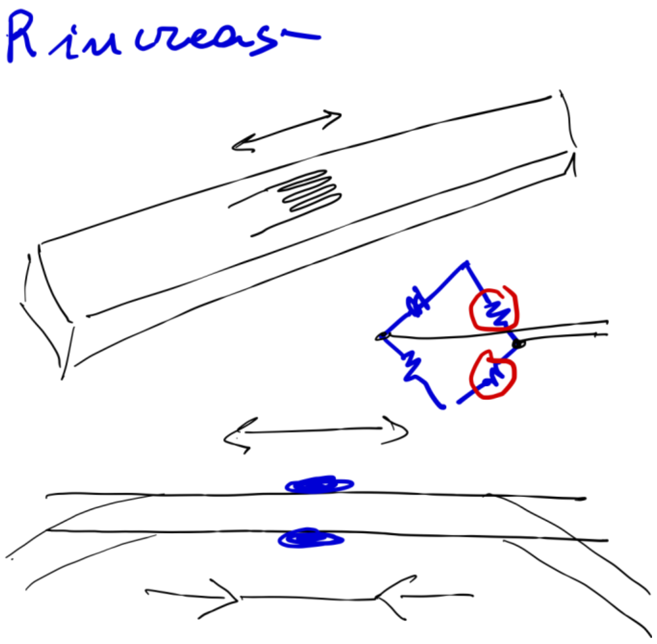
- Analog implementation
  - Multiplier or double-balanced mixer
    - Saturation
  - Passive filters difficult to design
  - Active filters easier to shape, but noisy
- Digital implementation
  - Saturation of the ADC
  - The low-pass filters integrate the signal in its time constant –> Numerical overflow

Example – Strain Gauge



Wikimedia, CC-BY-4.0 Cristian V, 2017



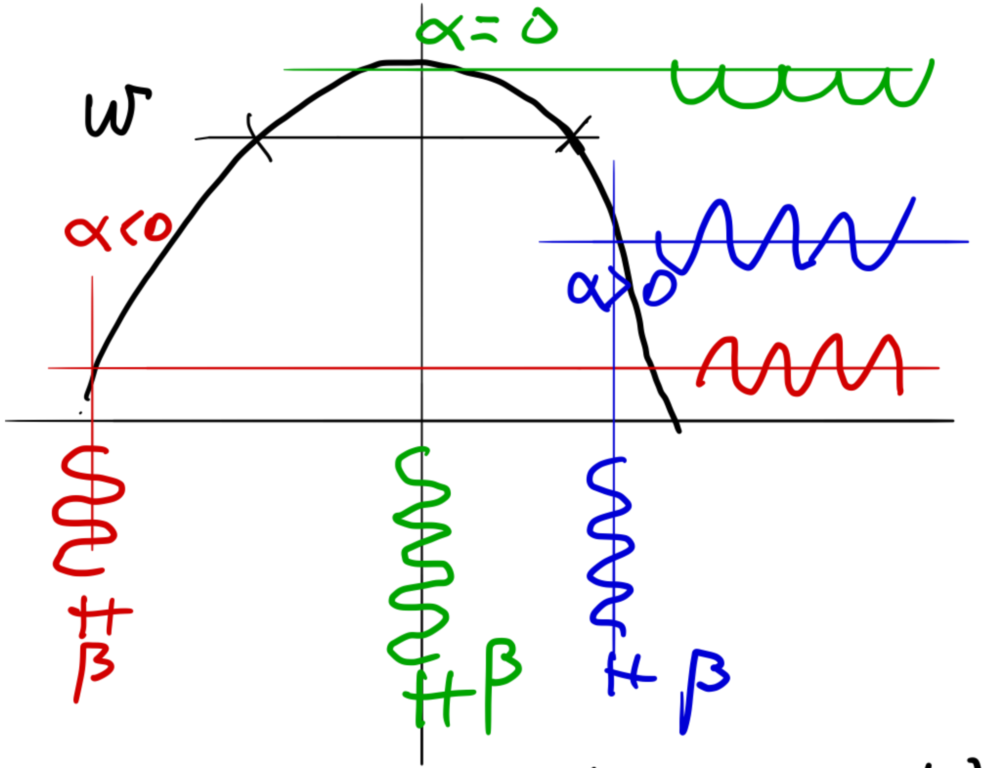


Tricks

- Thermal coefficient dR/RdT matches the material under test
    - Specific strain gauges for steel, concrete, Aluminum, etc.
    - Typical 1 ppm/K residual coefficient
- Beware of the glue
- Two-sensor symmetry doubles the gain and improves the stability
- Wheatstone bridge is magic
- 4-wires connection minimizes the effect of cable resistance
- Virtues of 600 Hz probe
    - multiple of 50 Hz and 60 Hz (EU/USA)
    - Notch filter cancels the pollution from power grid

Application – Spectroscopy

$$y = 1 - x^2/W$$



α signal (freq. offset)
β modulat. index
W 3 dB width

Add a picture with setup or block diagram

$$y = 1 - x^2/W \leftarrow x = \alpha + \beta\cos(\omega_m t)$$

$$y = 1 - \frac{1}{W}\left(\alpha + \beta\cos(\omega_m t)\right)^2$$

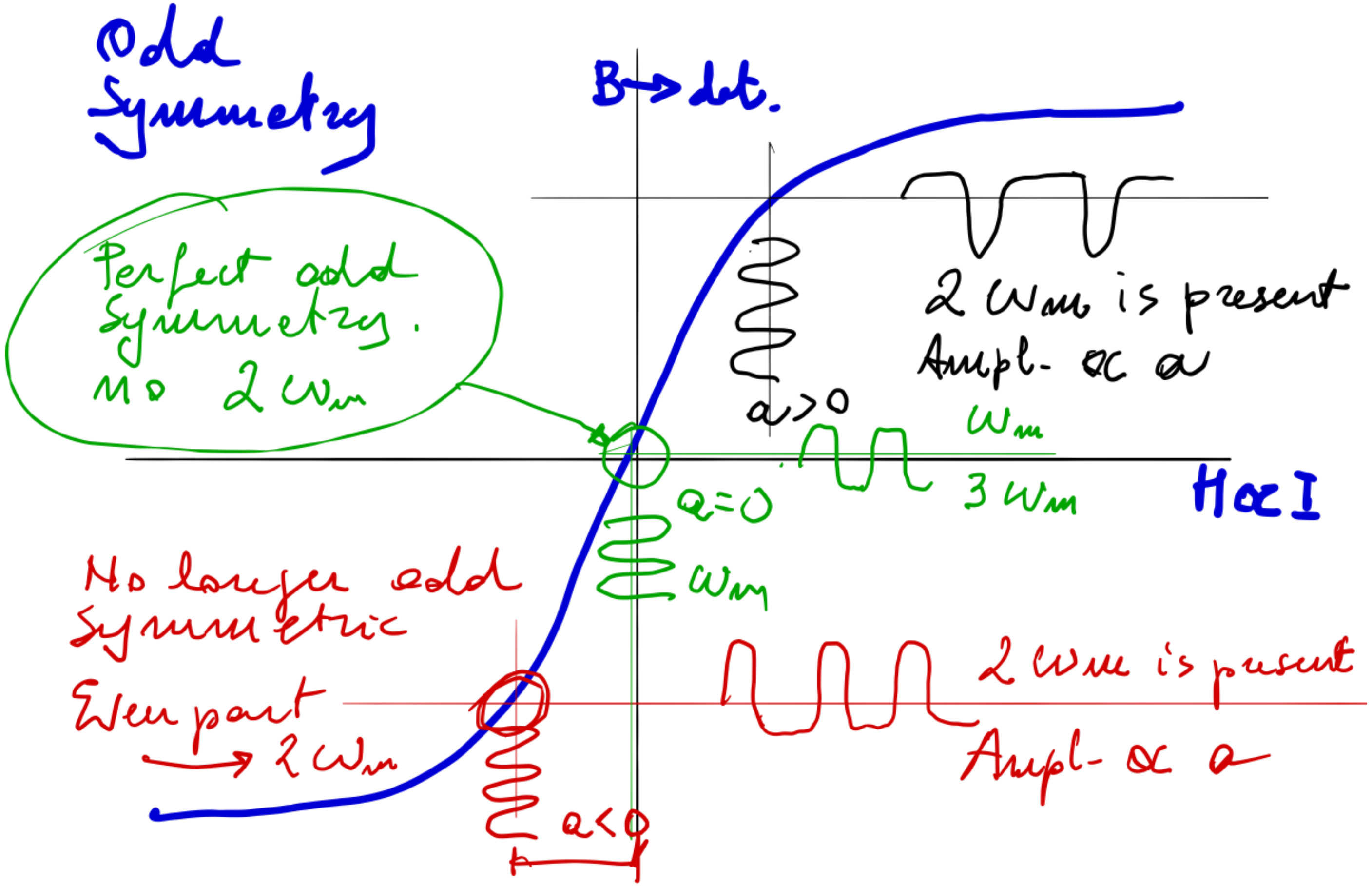
$$= 1 - \frac{1}{W}\left[\alpha^2 + 2\alpha\beta\cos(\omega_m t) + \beta^2\cos^2(\omega_m t)\right]$$

DC $\quad 1 - \frac{1}{W}\left(\alpha^2 + \frac{1}{2}\beta^2\right)$

$\omega_m \quad -2\frac{\alpha\beta}{W}\cos(\omega_m t)$ Signal

$2\omega_m \quad -\frac{1}{2}\frac{\beta}{W}\cos(2\omega_m t)$ Validation

Application – Magnetic Field



Example:
Magnetic Sat.

Detect
$3\omega_m$

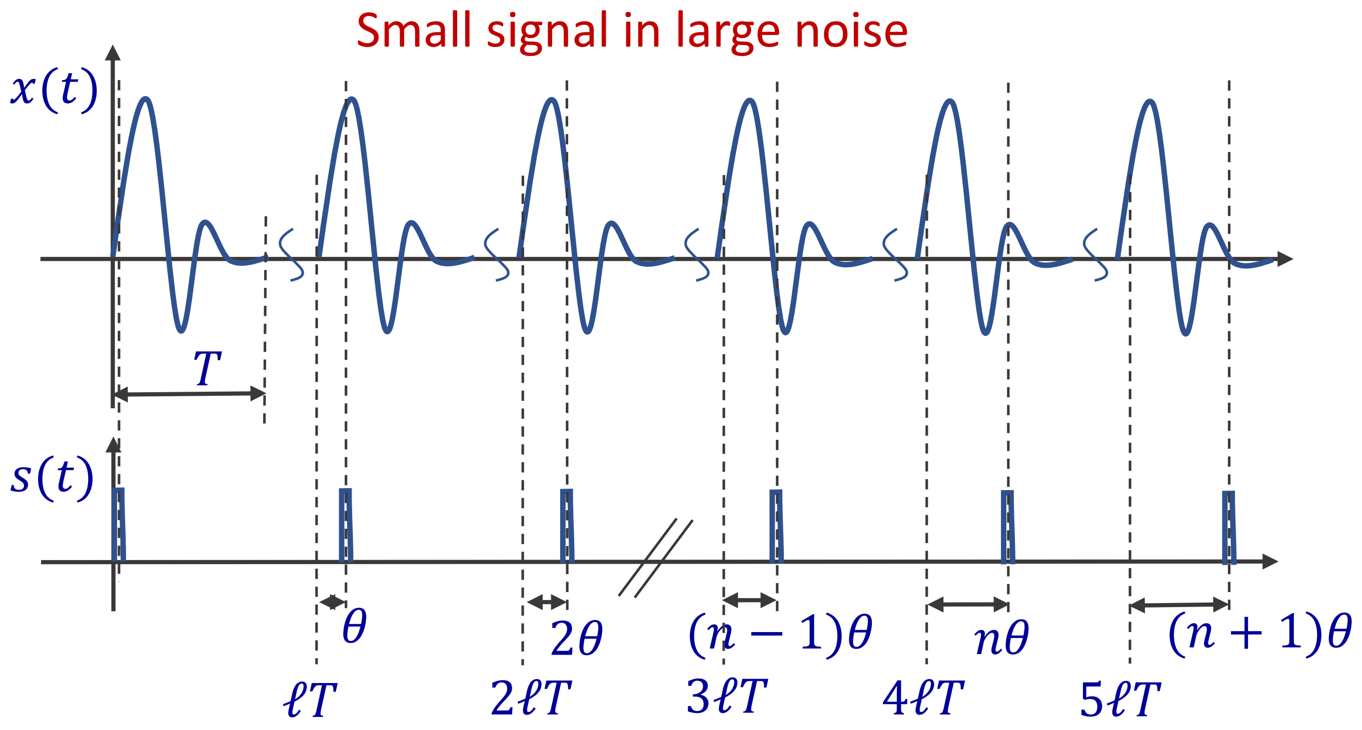
$$y = x^3 \leftarrow x = \alpha + \beta\cos(\omega t)$$

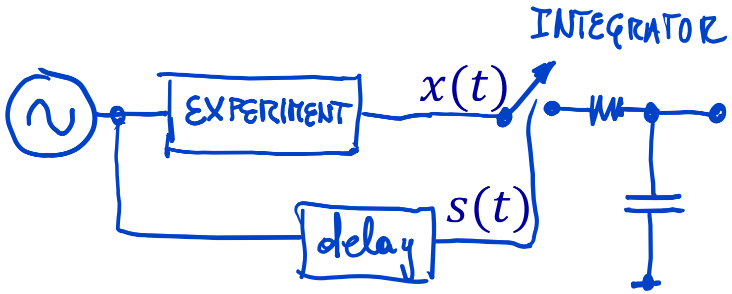
Use $\cos^3(\alpha) = \frac{3}{4}\cos(\alpha) + \frac{1}{4}\cos(3\alpha)$

DC	$\alpha^3 + \frac{3}{2}\alpha\beta^2$
ω	$\frac{12\alpha^2 + 3\beta^2}{4}\cos(\omega t)$
2ω	$\frac{3}{2}\alpha\beta^2\cos(2\omega t)$ signal
3ω	$\frac{1}{4}\beta^3\cos(3\omega t)$ Valid.

Boxcar Averager

Boxcar Averager





- Average on m samples for each $\tau = n\theta, n = 0 \ldots N$
- Takes $N + 1$ integrators
- The integer ℓ is a technical delay

Analog boxcar

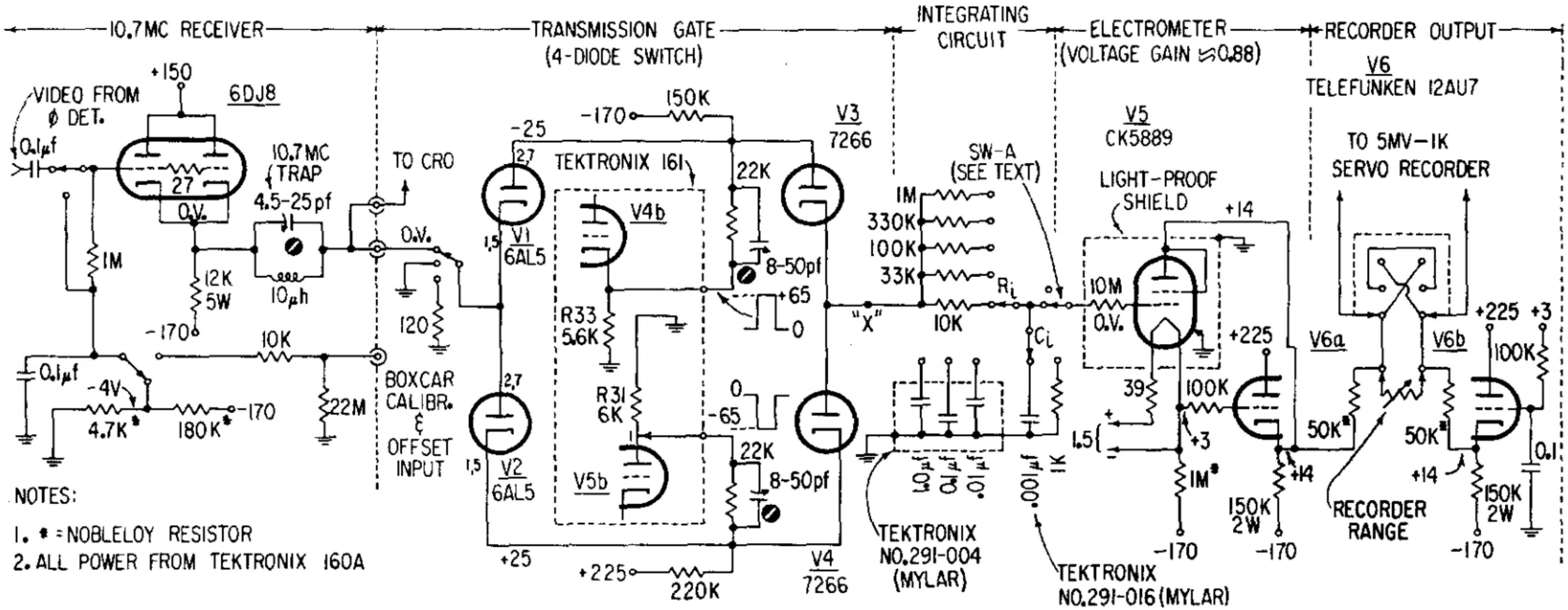
- Early 1950s
- Parallel –> multiple integrators
- Sequential –> one integrator, and slow recorder

Digital boxcar

- Fast electronics
- No need of delay, $\ell = 0$
- Needs large dynamic reserve
  - Use a fraction of ENoB
  - Integrator takes highe no of bits

A Sequential Boxcar in 1960

Blume, Fig. 1



R. J. Blume, `Boxcar' Integrator with Long Holding Times, Rev Scient Instrum 32(9) p.1016, Sept 1961

High-Resolution Time-To-Digital & Frequency-To-Digital Converters

Enrico Rubiola

CNRS FEMTO-ST Institute, Besancon, France

INRiM, Torino, Italy

Outline

- Basic counters (RF & microwave)
- The input trigger
- Clock interpolation techniques
- Π, Λ and Ω counter, and statistics

Updated March 6, 2023

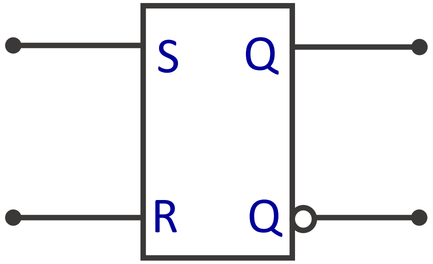
Excerpt from Counters.pptx

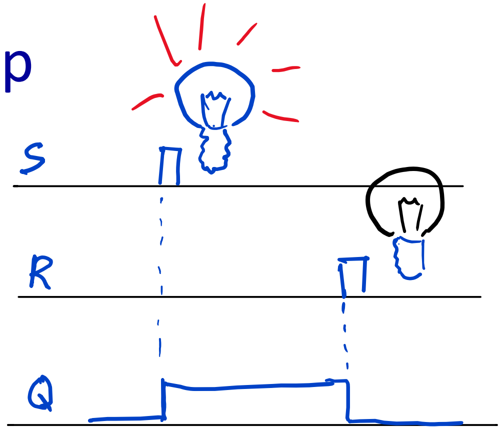
home page http://rubiola.org

1 – Basic TDCs and FDCs

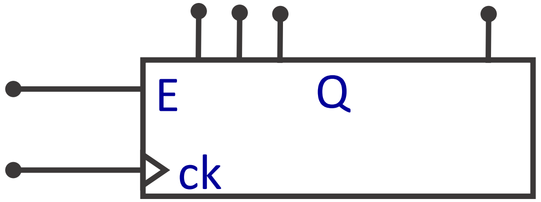
Digital hardware

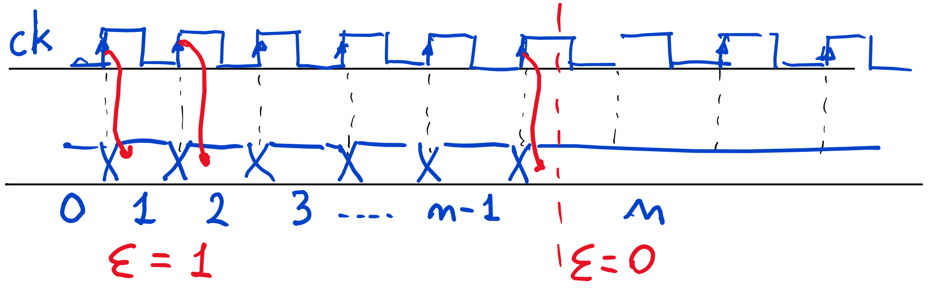
Set-Reset Flip-Flop



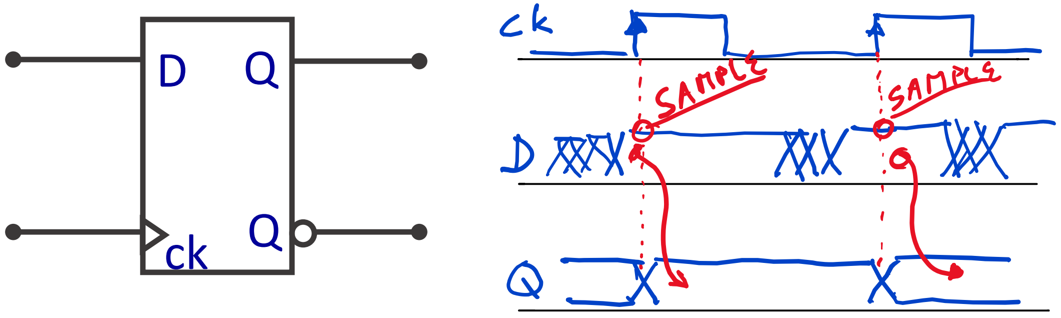


Binary counter

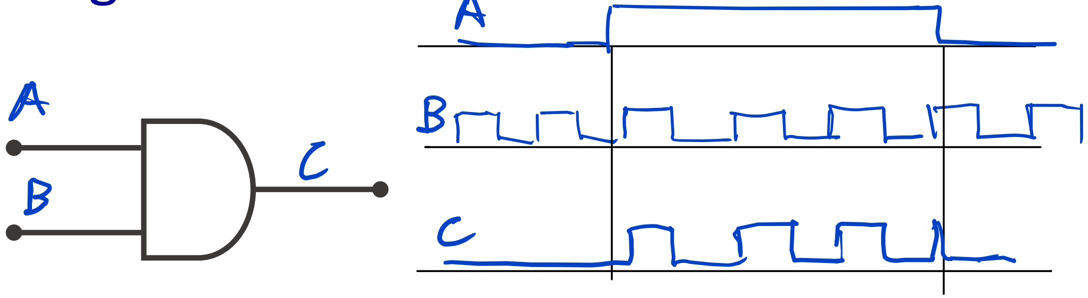




D-Type Flip-Flop (digital sampler)



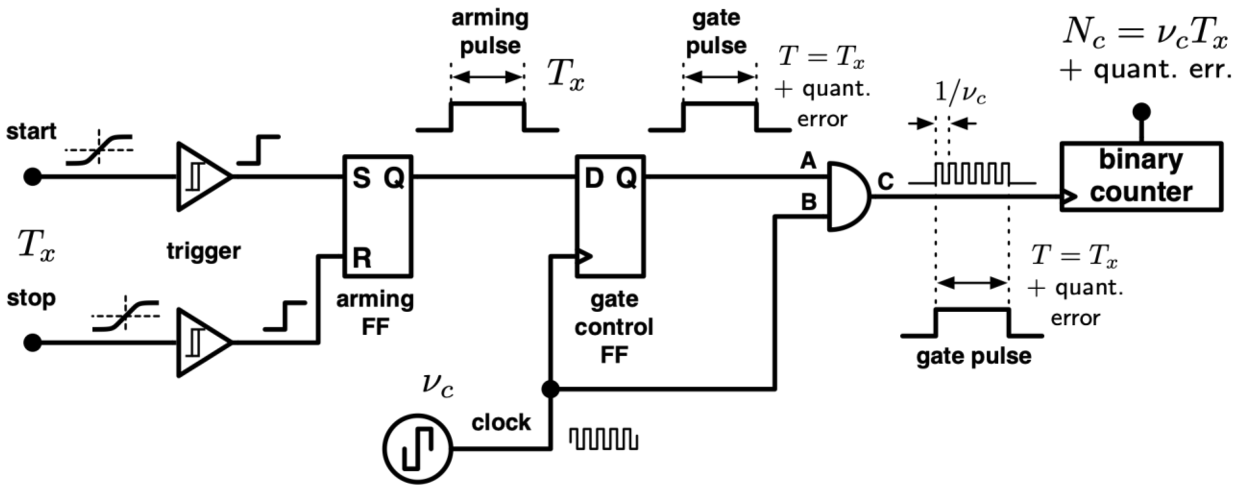
And gate

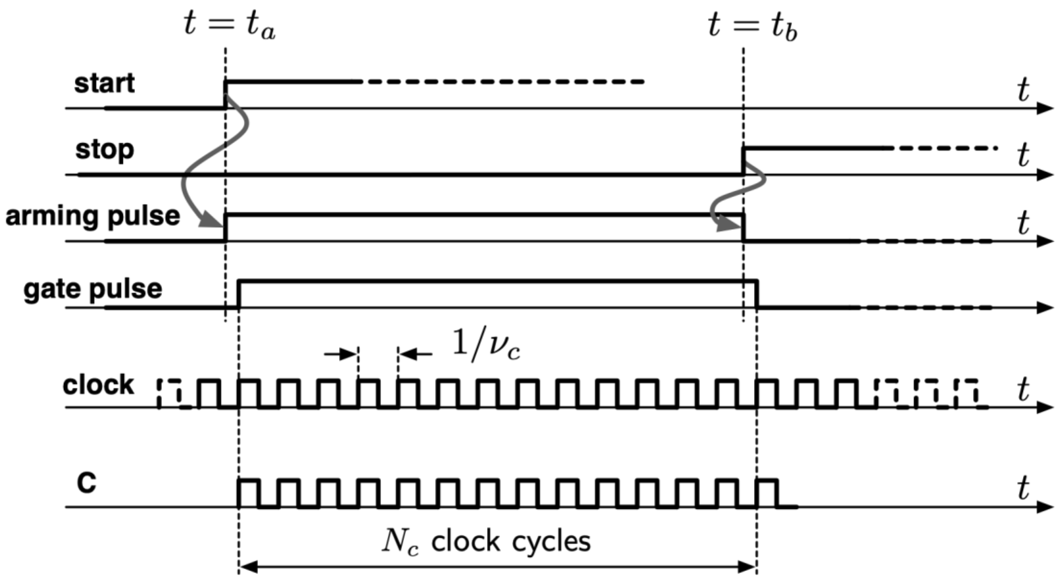


1 & 1 => 1

0 => 0

Time interval



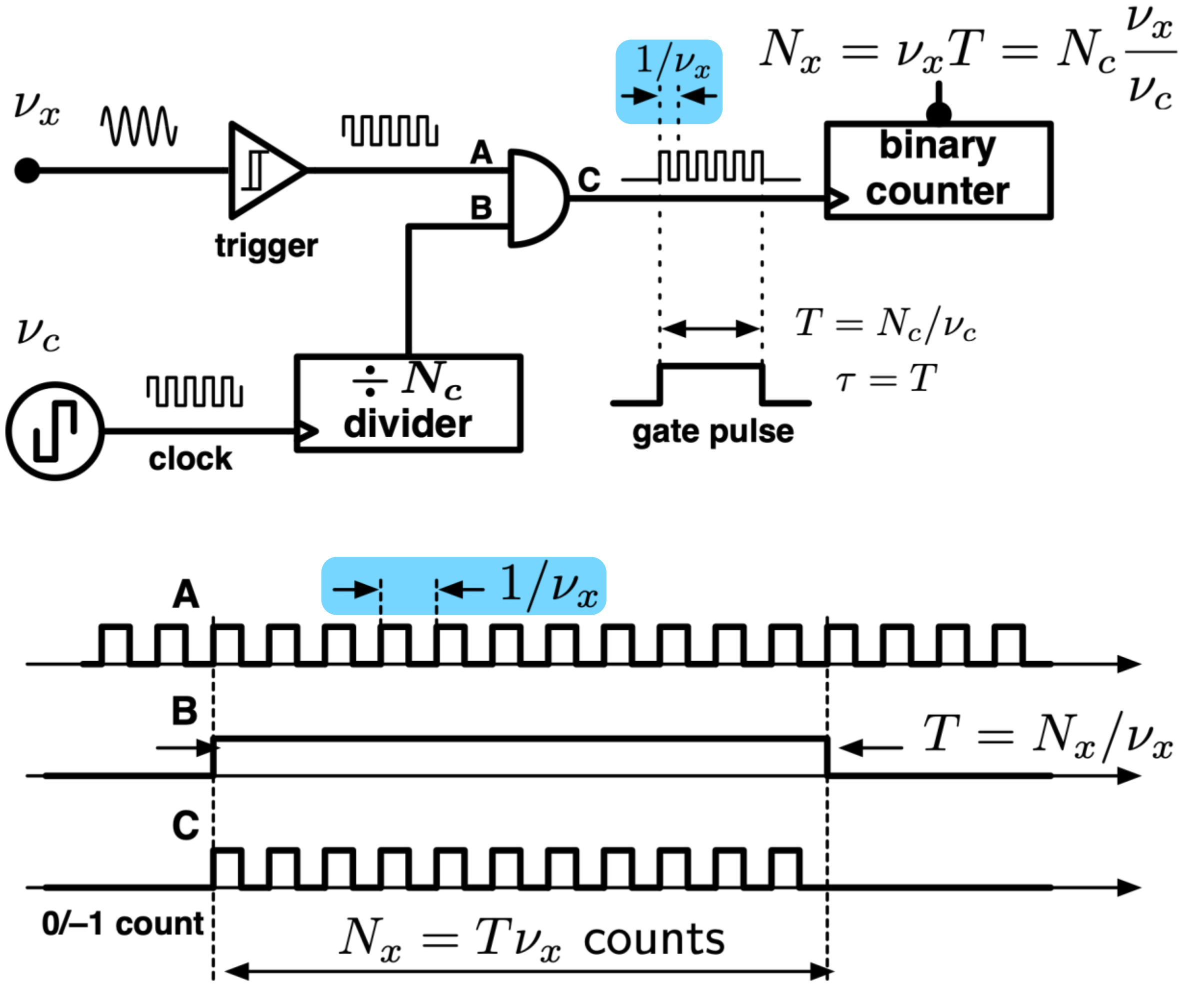


The gate control FF is a trick to synchronize the inputs to the clock

The resolution is set by the clock period $1/\nu_c$

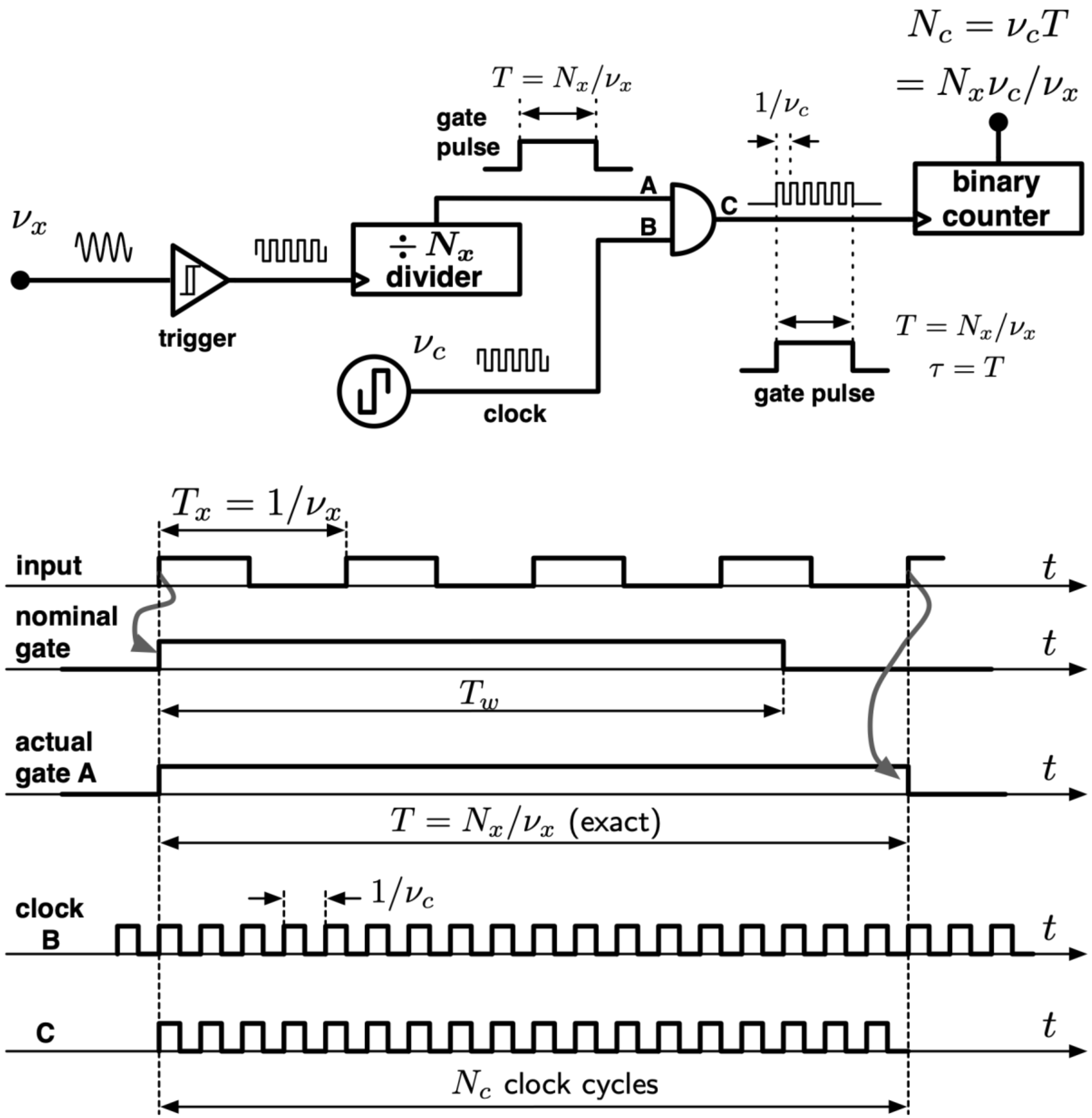
The (old) frequency counter

The gate-control FF is not shown



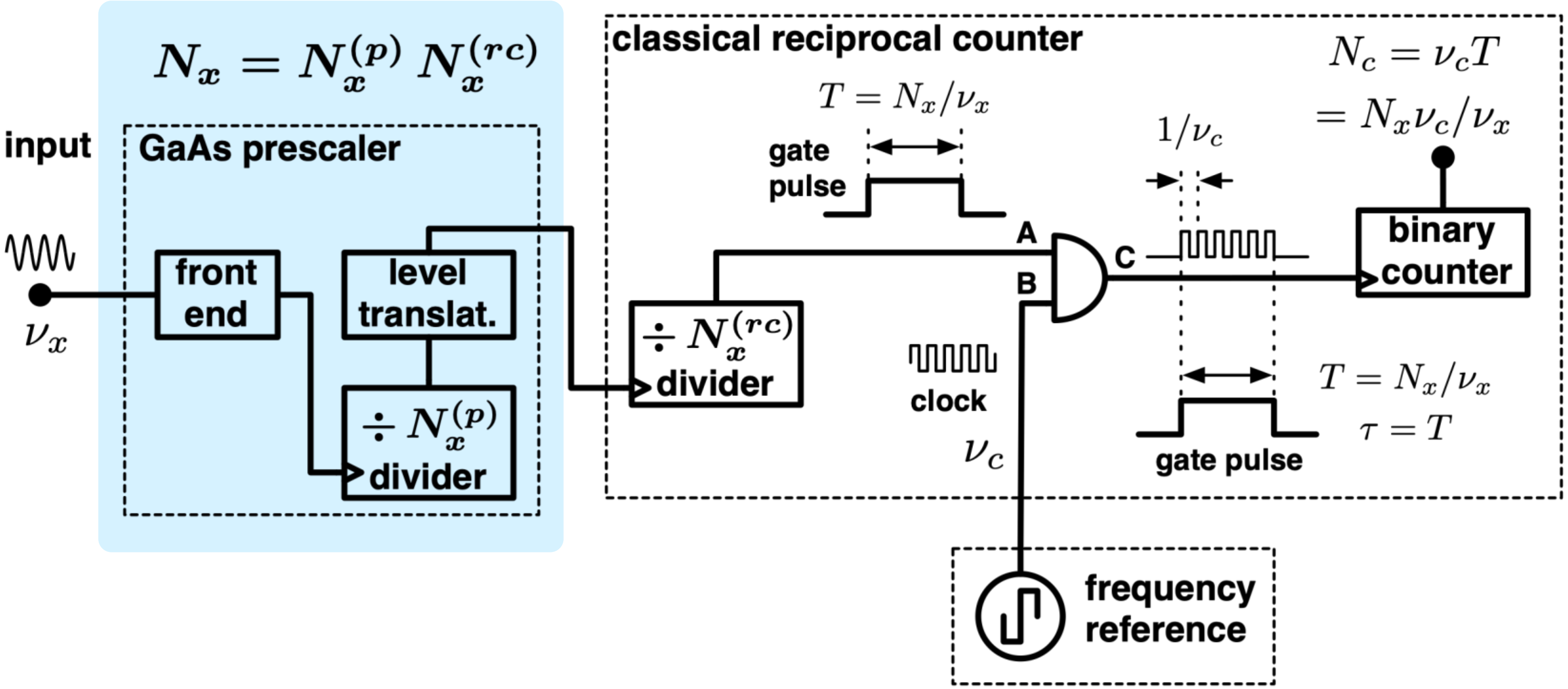
The resolution is set by the input period $1/\nu_x$, which can be poor

Classical reciprocal counter



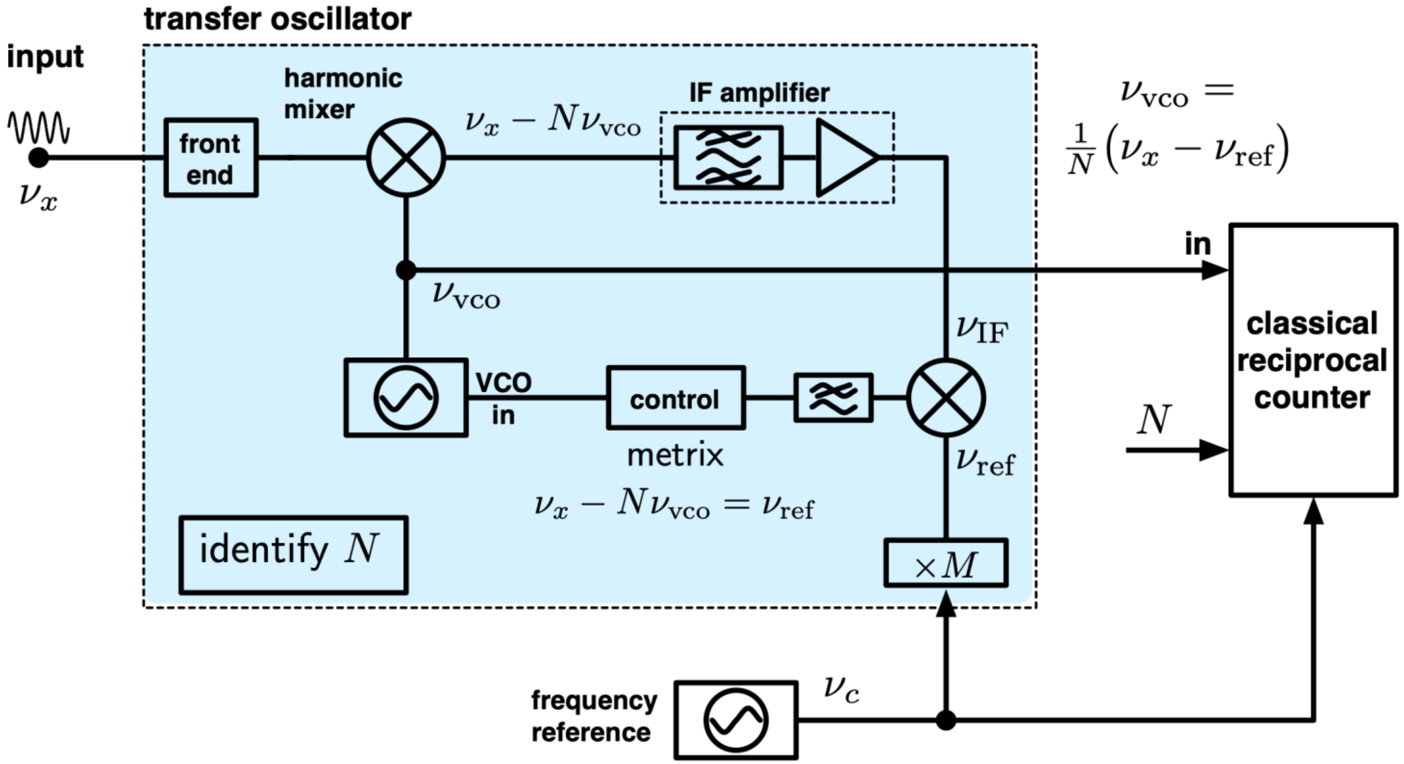
- Use the highest clock frequency permitted by the hardware
- The measurement time is a multiple of the input period

Prescaler



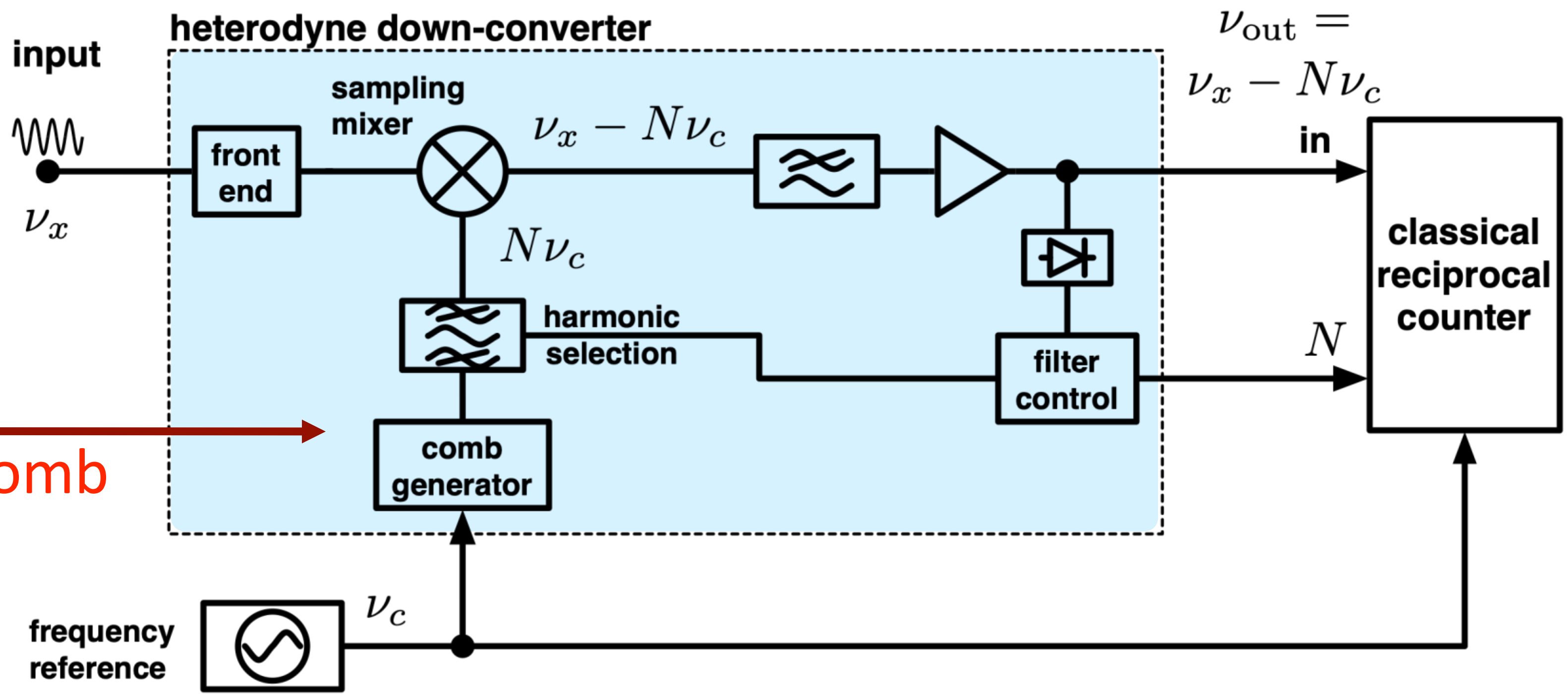
- The prescaler is a n-bit binary divider ÷ 2^n (decimal scalers are gone)
- GaAs dividers work up to at least 20 GHz
- Reciprocal counter => there is no resolution reduction
- Most microwave counters use the prescaler

Transfer oscillator



- The transfer oscillator is a PLL
- Harmonics generation takes place inside the mixer
- Harmonics locking condition: $N\nu_{vco} = \nu_x$
- Frequency modulation Δf is used to identify N
- Rather complex scheme,
  $\times N \quad => \quad \Delta\nu \; N\Delta\nu$

Heterodyne counter



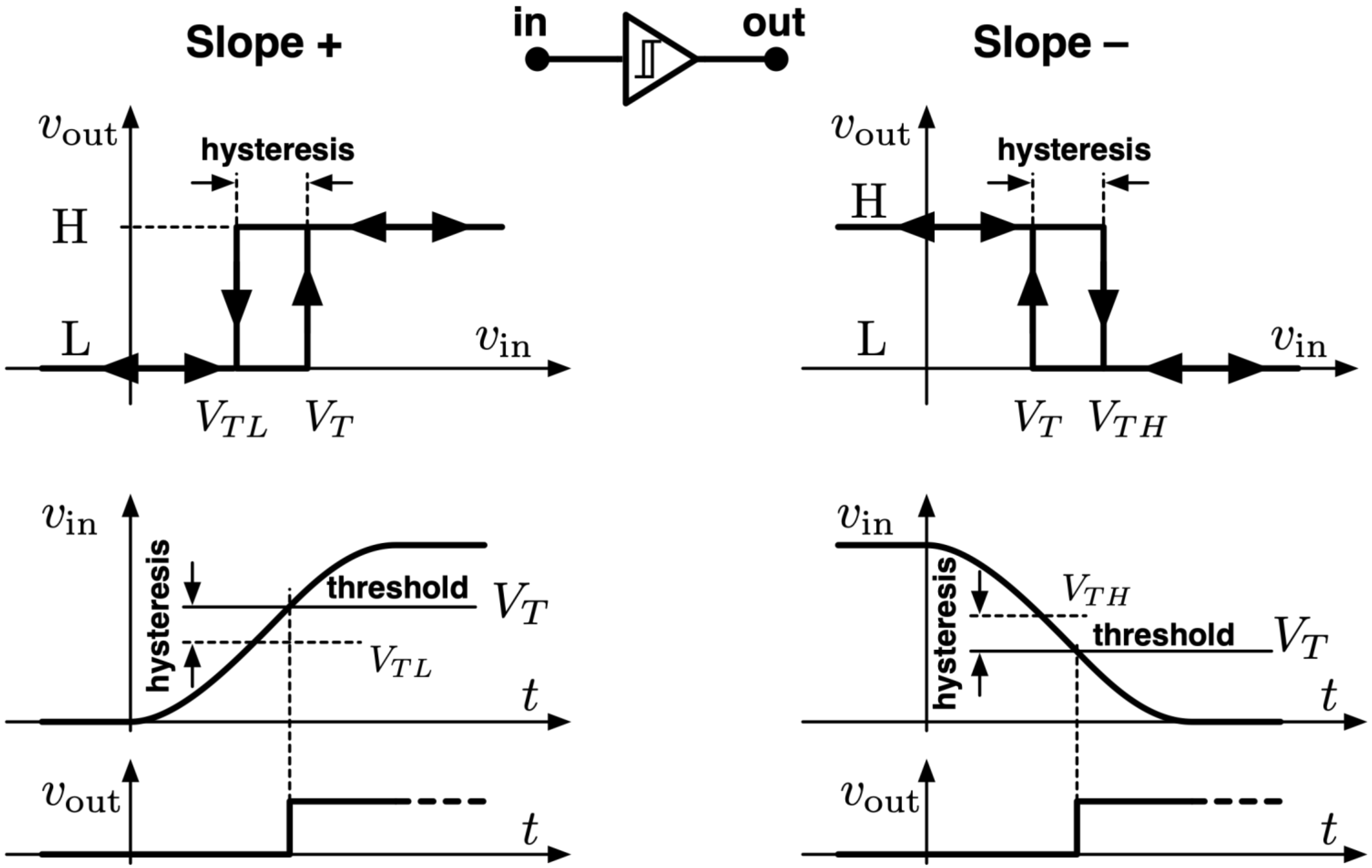
now,

FS Comb

- Down-conversion: $f_b = |\nu_x - N\,\nu_c|$
- ν_b is in the range of a classical counter (100–200 MHz max)
- no resolution reduction in the case of a classical *frequency* counter (no need of reciprocal counter)
- Old scheme, nowadays used only in some special cases (laser frequency metrology)

2 – Trigger

Trigger hysteresis



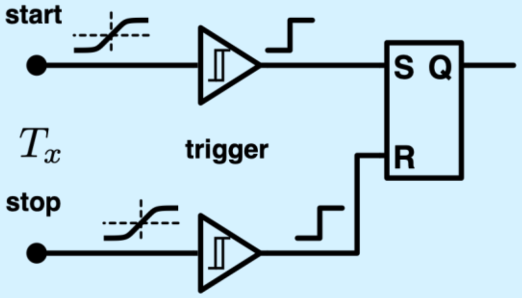
Hysteresis is necessary to avoid chatter in the presence of noise

Threshold fluctuation

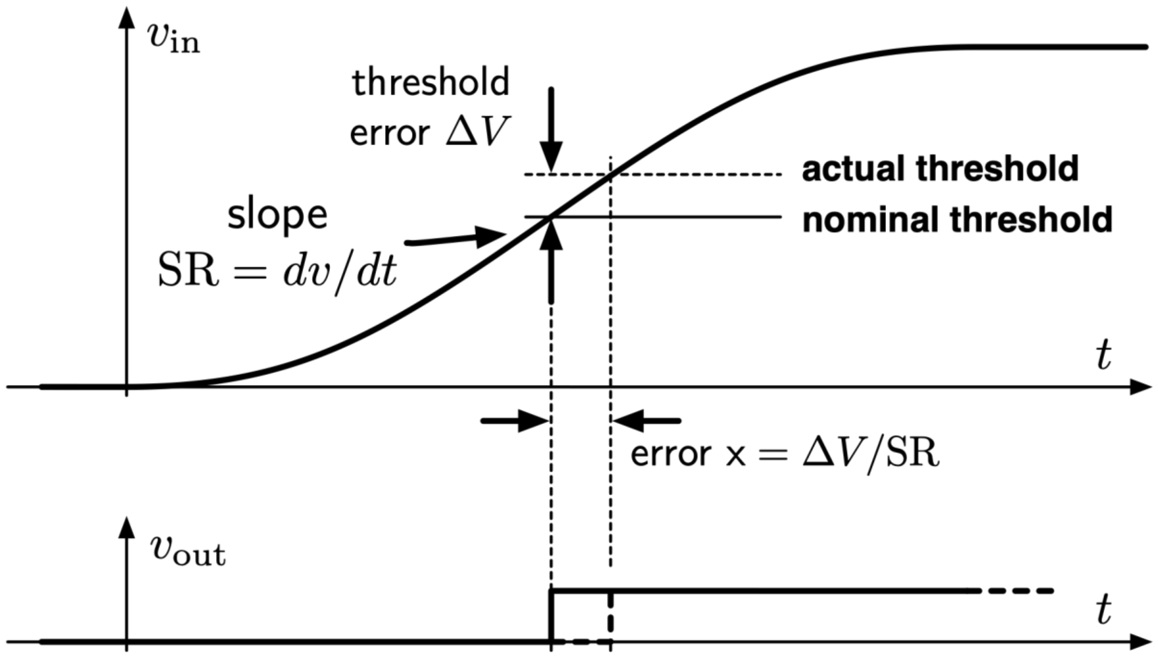
‘stop’ − ‘start’

systematic
$$\mathsf{x} = \frac{(\Delta V)_b}{\mathrm{SR}_b} - \frac{(\Delta V)_a}{\mathrm{SR}_a}$$

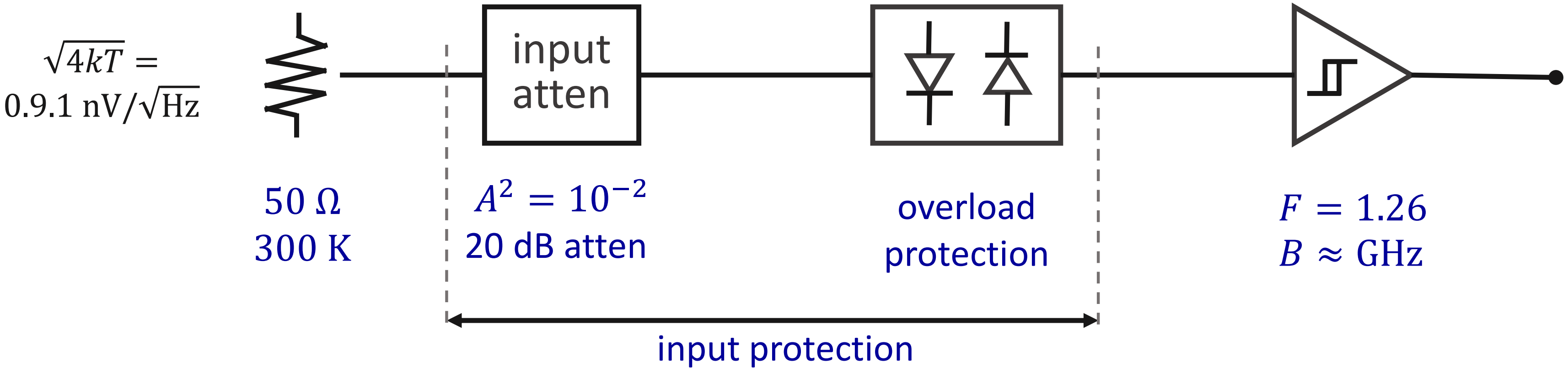
random
$$\sigma_{\mathsf{x}}^2 = \frac{(\sigma_V^2)_a}{\mathrm{SR}_a^2} + \frac{(\sigma_V^2)_b}{\mathrm{SR}_b^2}$$



Threshold fluctuation



Don’t blame the trigger



$\sqrt{4kT} =$ 0.9.1 nV/$\sqrt{\text{Hz}}$

50 Ω
300 K

input
atten

$A^2 = 10^{-2}$
20 dB atten

overload
protection

$F = 1.26$
$B \approx$ GHz

input protection

Input noise $\sqrt{4kTB}$ of frequency counters			
Type	max freq	Noise BW	e_n
HP 5370	225 MHz	900 MHz	27 µV
SR 620	1.3 GHz	5.2 GHz	66 µV
thumb rule: (noise BW) = 4 (max input f)			

Account for 20 dB loss and noise factor 126 (1 dB)
Equivalent noise figure $F = 126$ (21 dB)

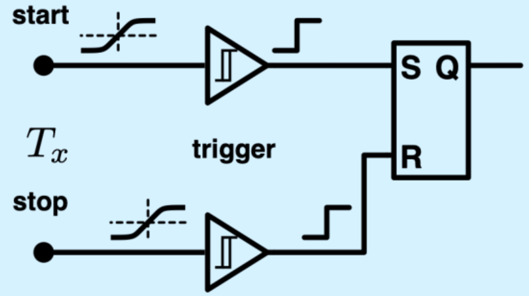
Total RMS noise	
HP 5370	306 µV
SR 620	736 µV

Trigger noise – oversimplified

'stop' − 'start'

systematic $$x = \frac{(\Delta V)_b}{\mathrm{SR}_b} - \frac{(\Delta V)_a}{\mathrm{SR}_a}$$

random $$\sigma_x^2 = \frac{(\sigma_V^2)_a}{\mathrm{SR}_a^2} + \frac{(\sigma_V^2)_b}{\mathrm{SR}_b^2}$$



SR = Slew Rate (slope)

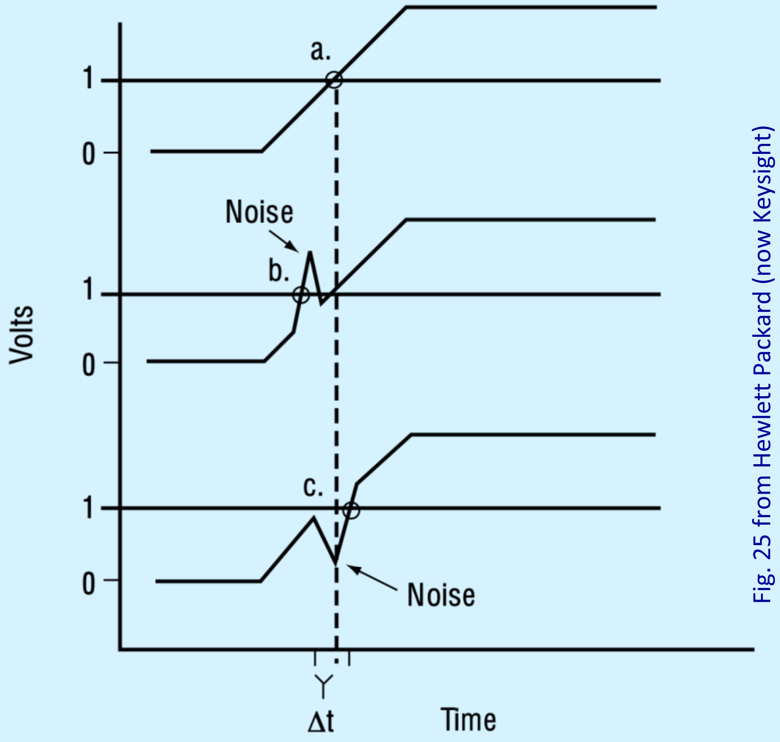


Fig. 25 from Hewlett Packard (now Keysight)
Application Note an200-3, 1997

- The effect of noise is often explained with a plot like this
- Yet, the formula holds in the absence of spikes!!!
- To the general practitioner, this explanation looks simple

See also:
E. Rubiola & al., Proc. 46
FCS pp.265-269, May 1992

Trigger behavior vs bandwidth

Noise rms slope

$$\mathrm{SR}_n^2 = 4\pi^2 \int_0^B f^2 S_V(f)\, df$$

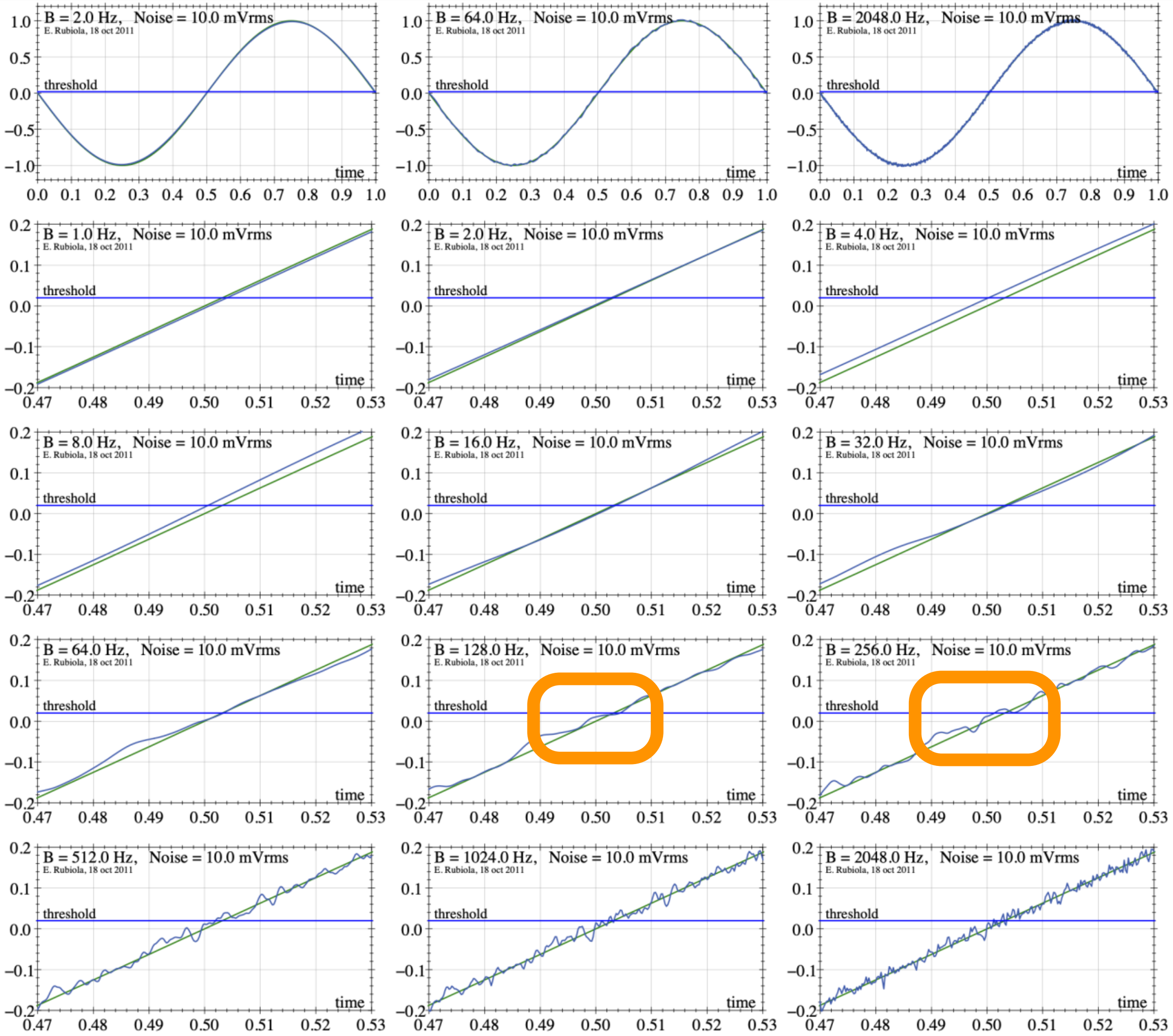
$$\mathrm{SR}_n^2 = \frac{4\pi^2}{3} \sigma_V^2 B^2$$

Critical slope

$$\mathrm{SR}_s^2 = \frac{4\pi^2}{3} S_V B^3$$

$$\mathrm{SR}_s^2 = \frac{4\pi^2}{3} \sigma_V^2 B^2$$

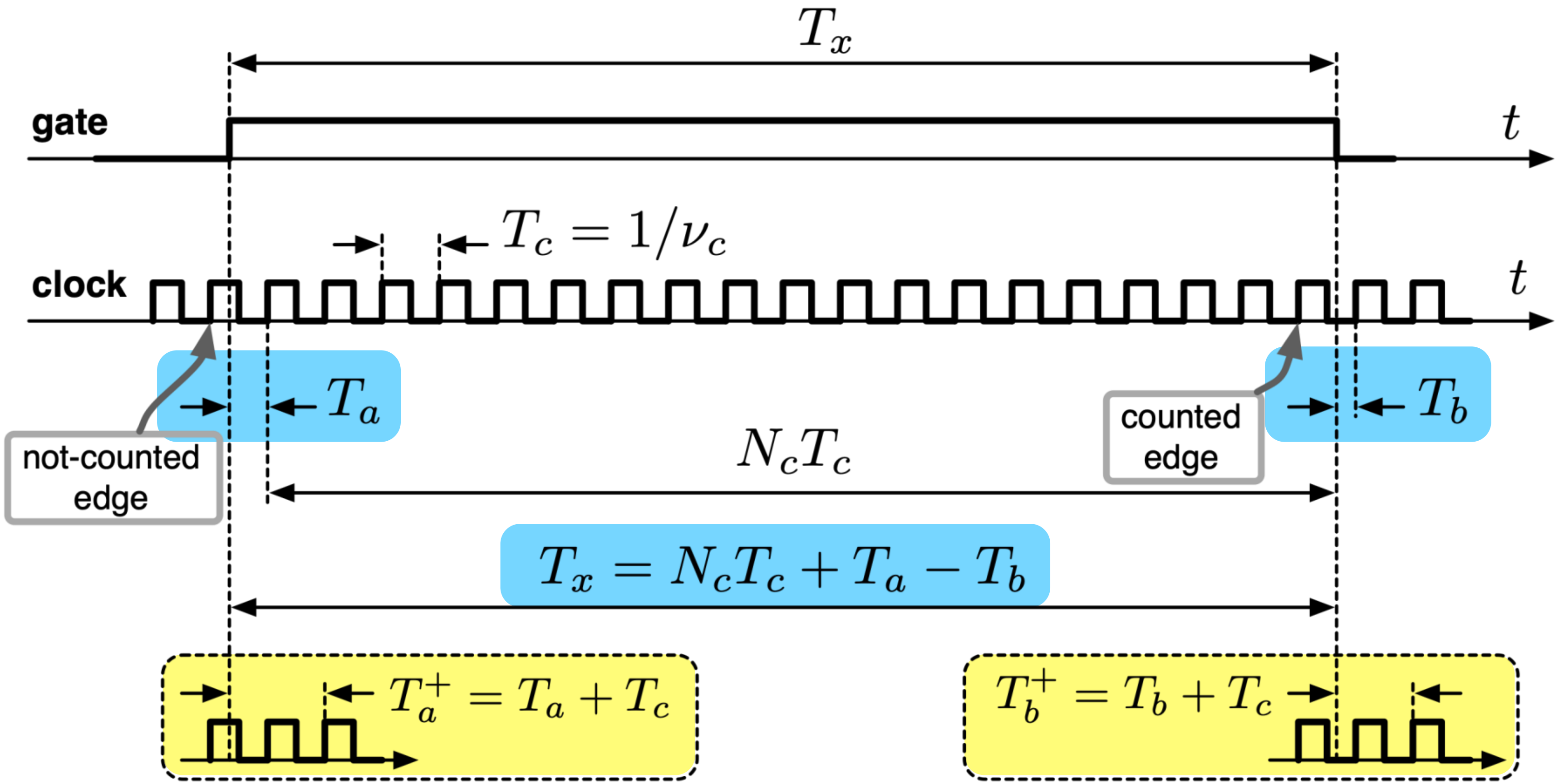
Signal slope equals rms noise slope



- When the noise slope exceeds the clean-signal slope, the total slope changes sign
- There result spikes, and systematic lead error

3 – Interpolation Schemes

Clock interpolation

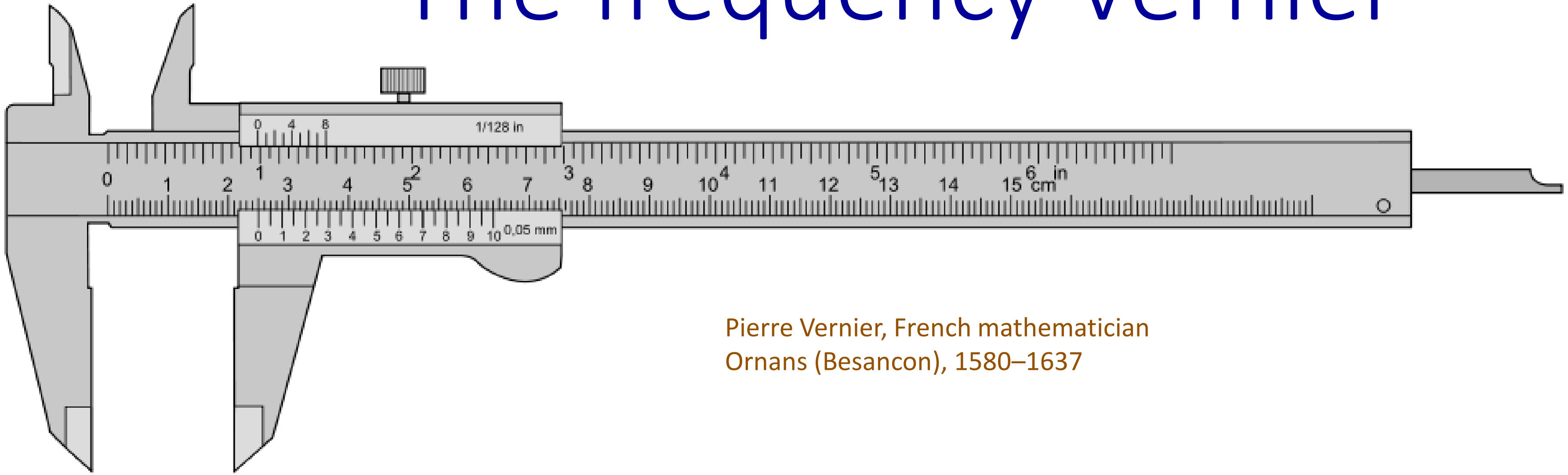


Too short T_a and T_b are difficult to measure, so we add one T_c to each

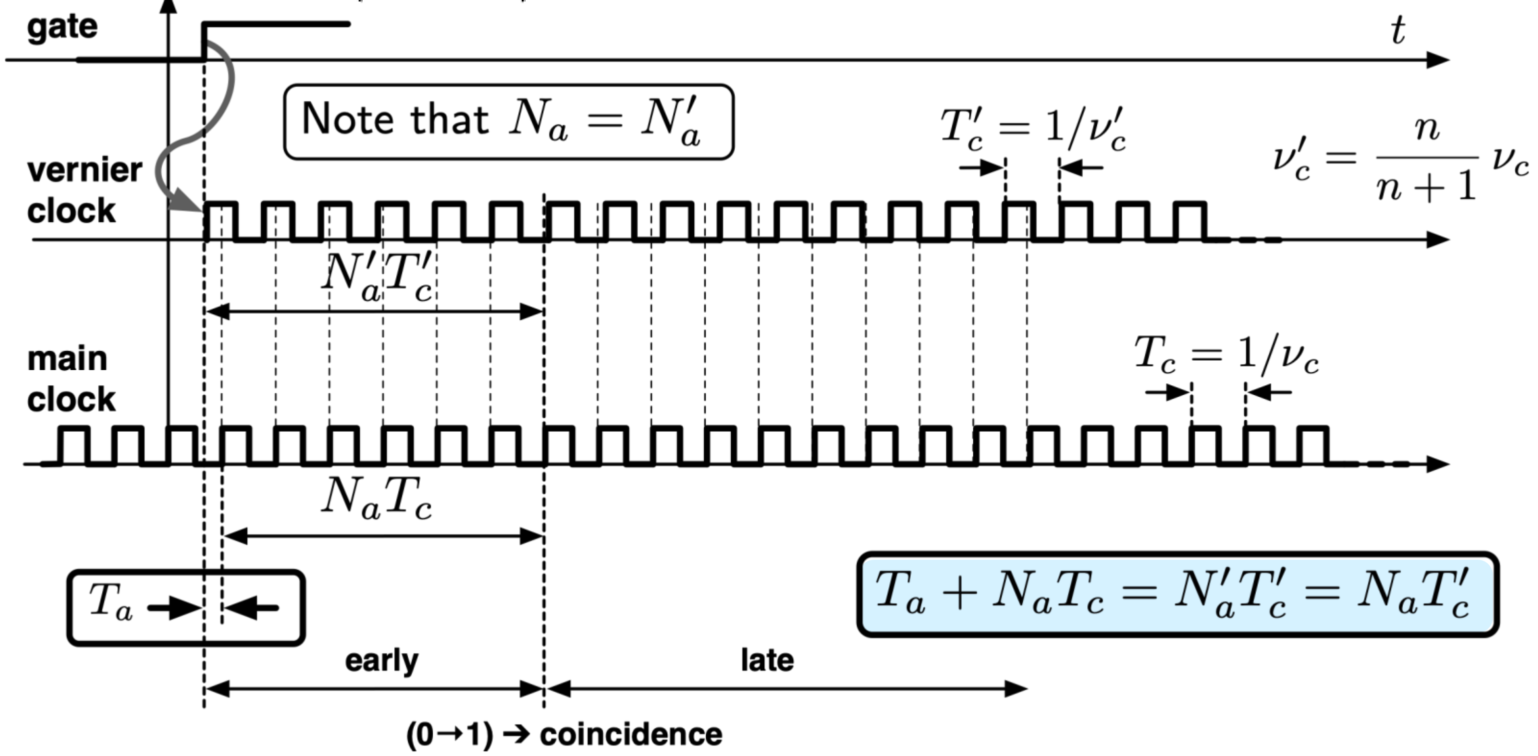
Interpolation is made possible by the fact that
the clock frequency is constant and accurately known

The frequency Vernier

Picture Lucasbosch, CC-BY-SA-3.0



Pierre Vernier, French mathematician
Ornans (Besancon), 1580–1637



Example (HP5370A)

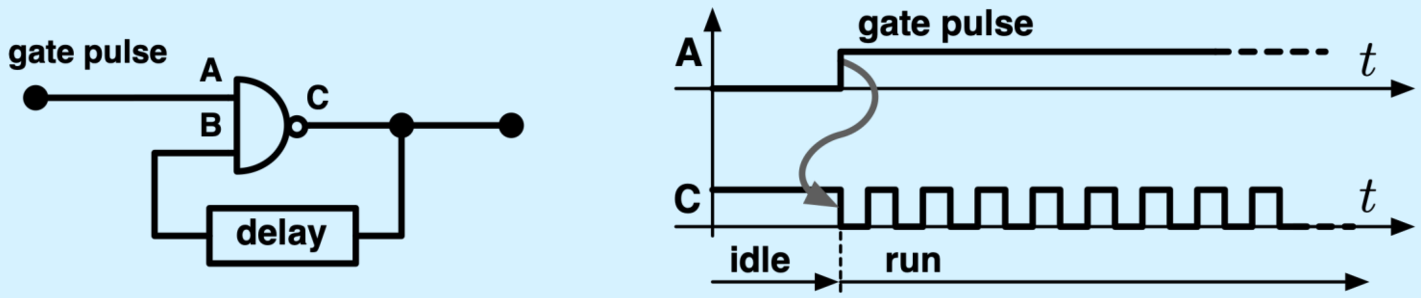
$f_c = 200$ MHz (5 ns)

$n = 256$

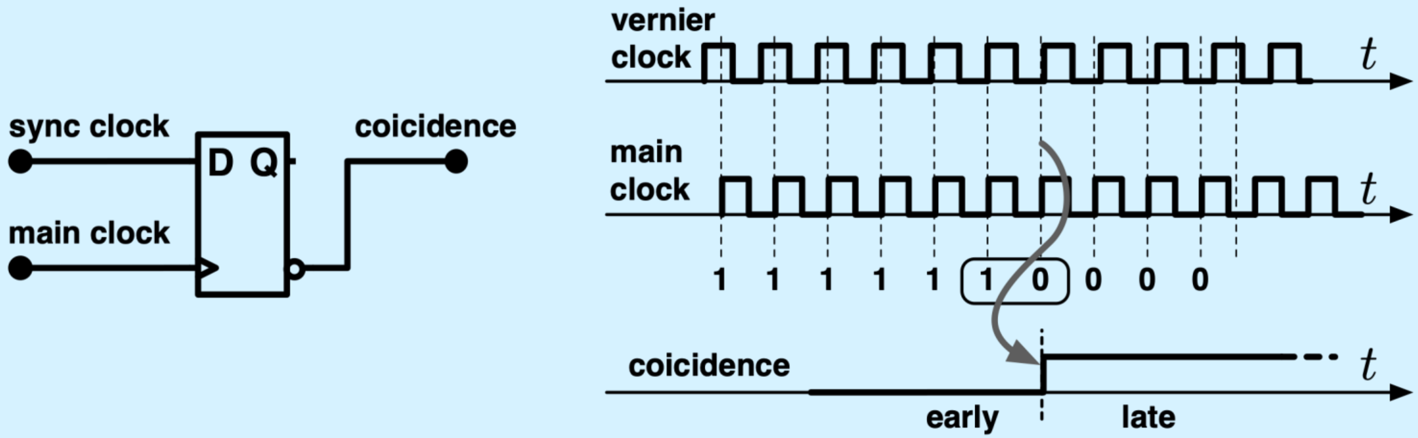
$1/257 \times 5$ ns $= 20$ ps

The key elements

Synchronized oscillator



Coincidence detector



Example: Hewlett Packard 5370A

Clock $f_c = 200$ MHz $\Rightarrow$ $\delta T_x = 5$ ns (ECL technology)

Vernier $n = 256$ $\delta T_a = \delta T_b = \frac{1}{256}\delta T_x = 19.5$ ps

It takes max 257 cycles of f_c for the two clocks to coincide

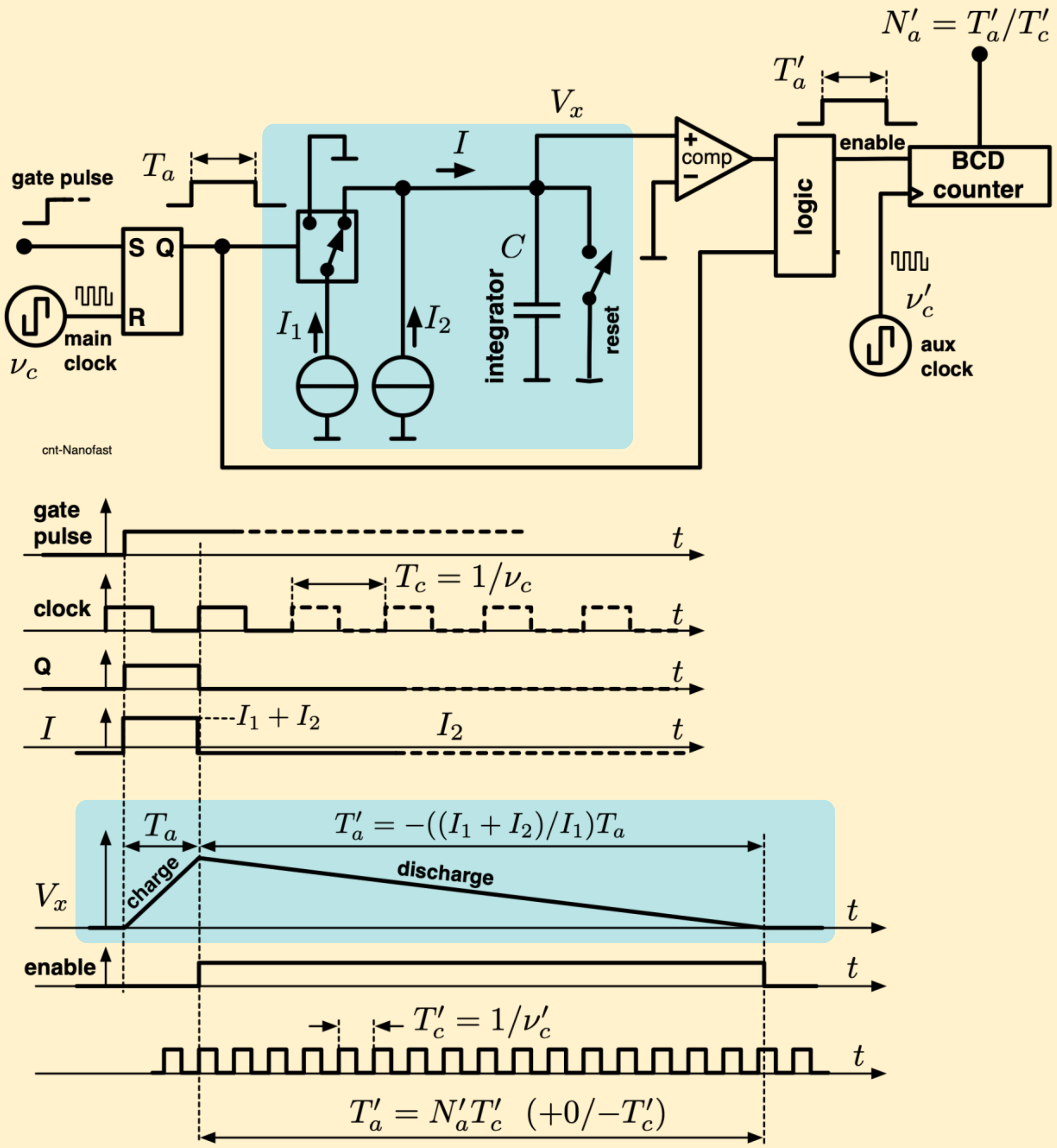
Conversion time $T = nT_c = 1.283$ µs

Resolution

free space, $\delta\ell = c\delta T_a = 6$ mm

cable, $v = 0.67\ c$, $\delta\ell = 4$ mm

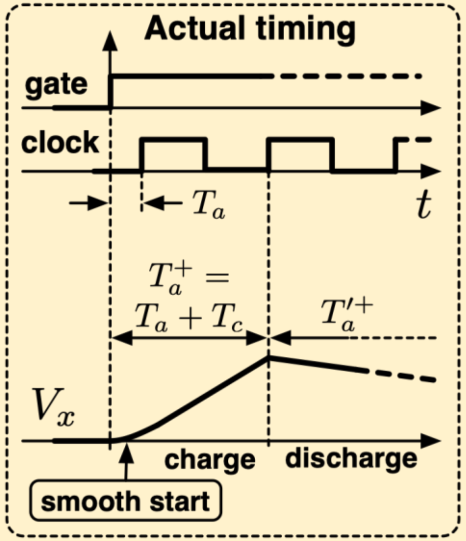
The Nutt’s dual-slope interpolator



Similar to the dual-slope voltmeter

R. Nutt, Digital time intervalometer, RSI 39(9) p.1342-1345, sep 1968

Example
Nanofast 536 B (early 1970s!)
Smithsonian Astrophysical Lab
$f_c = $ 20 MHz —> $T_c = $ 50 ns
$(I_1 + I_2)/I_2 = 4096$
$T_c/[(I_1 + I_2)/I_2] = $ 12 ns



Example: Nanofast 536 B

Smithsonian Astrophysical Laboratory

Main clock $f_c = 10$ MHz → $\delta T = T_c = 100$ ns

Time Interval amplifier $\frac{I_1}{I_2} = 4000$

$T'_a \in (200\,\mu s, 400\,\mu s)$

aux. clock 20 MHz for the measurement of T'_a

$\delta T'_a = T'_c = 50$ ns (1/20 MHz)

$\delta T_a = \frac{I_2}{I_1} T'_c$ $\delta T_a = \frac{1}{4000} \times 50$ ns $= 12.5$ ps

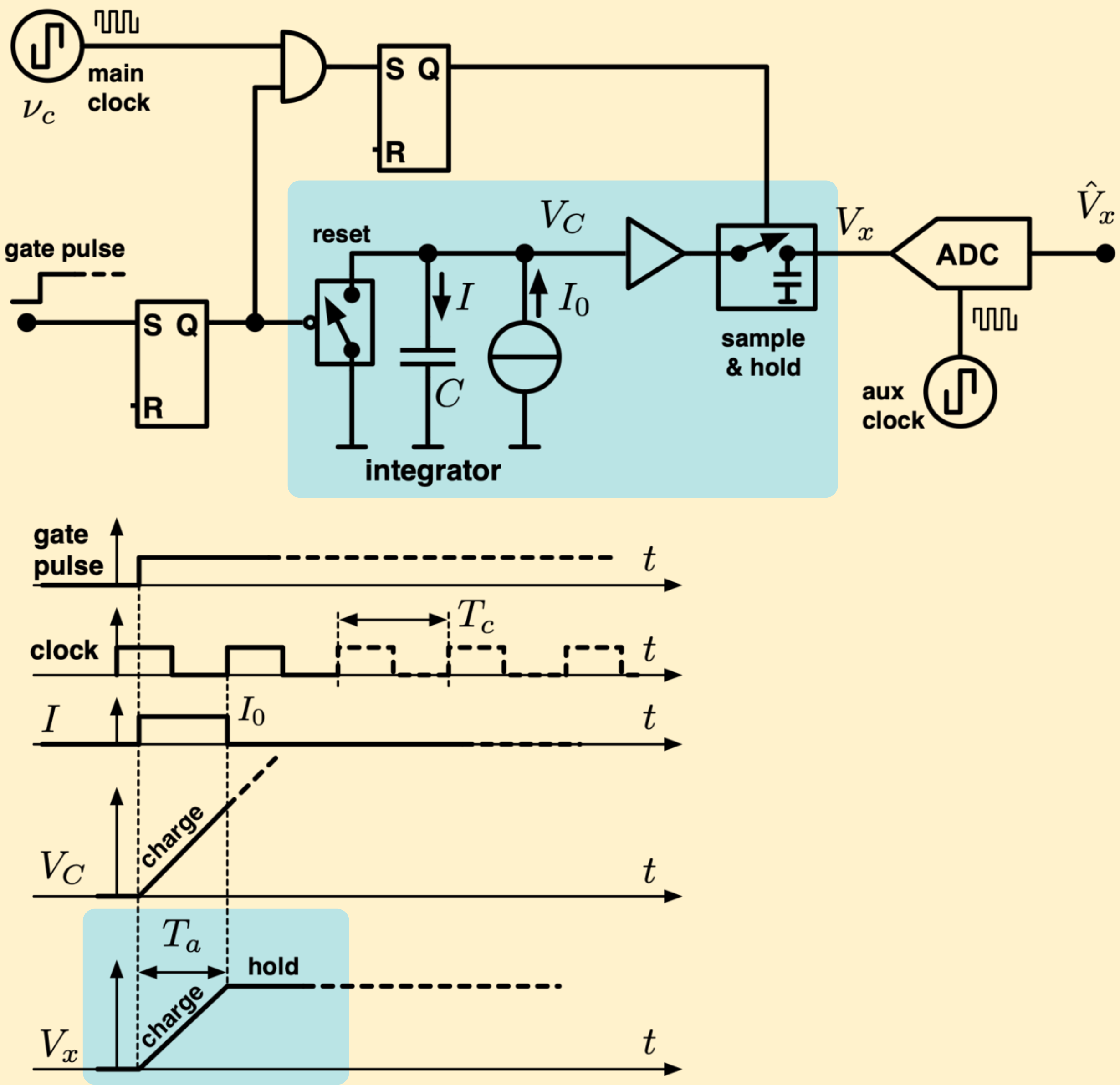
The Nanofast 536 B counter is (was?) a part of the Mark IV system for Very Long Baseline Interferometry (VLBI).

Early TTL technology

Note: a pulse propagates in a cable at $c' \approx \frac{2}{3} c$

δT_a is equivalent to a length of 2.5 mm

The ramp interpolator

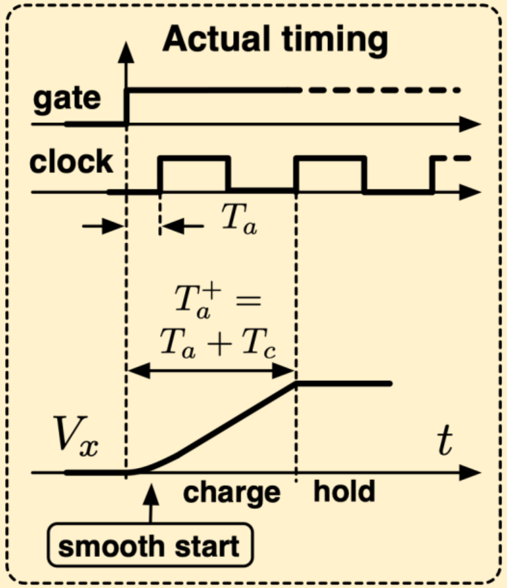


Example (Stanford SR620)

f_c = 90 MHz (T_c = 11.1 ns)

11 bits

$T_c / 2^{11}$ = 5.4 ps



This costs 1 bit ADC resolution loss

Example: Stanford SR 620

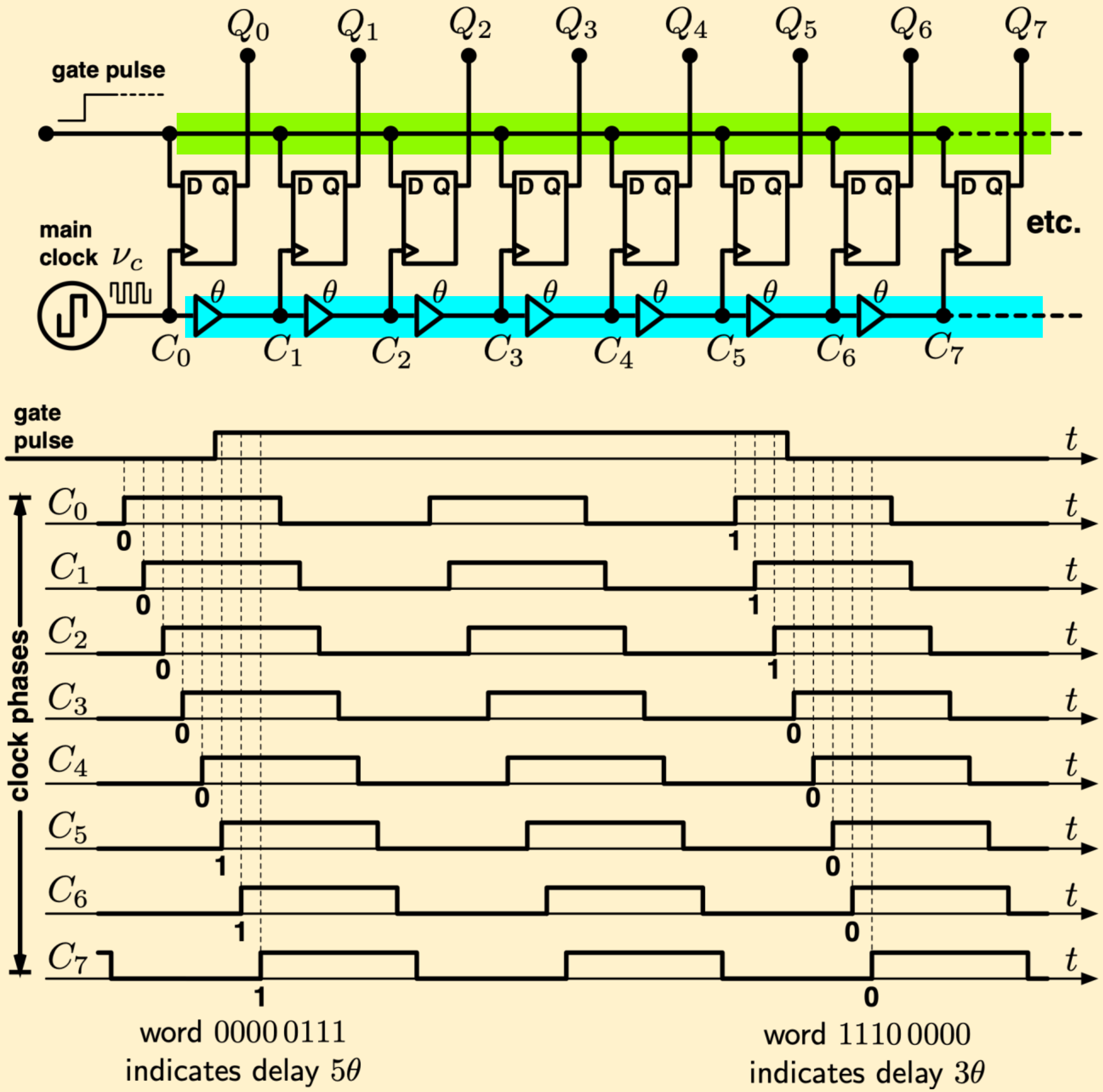
f_c = 90 MHz   phase-locked to the 10 MHz reference.

T_c = 11.1 ns   ECL Technology

12 bit converter, 1 bit lost because of the extra T_c

11 bits $\delta T_q = \frac{11.1\,\text{ns}}{2^{11}} = 5.4\,\text{ps}$

Thermometer-code interpolator



Also called Multi-tapped delay-line interpolator

FPGA implementation

- Needs full layout control
- The pipeline may not fit in a cell

Great for ASIC implementation

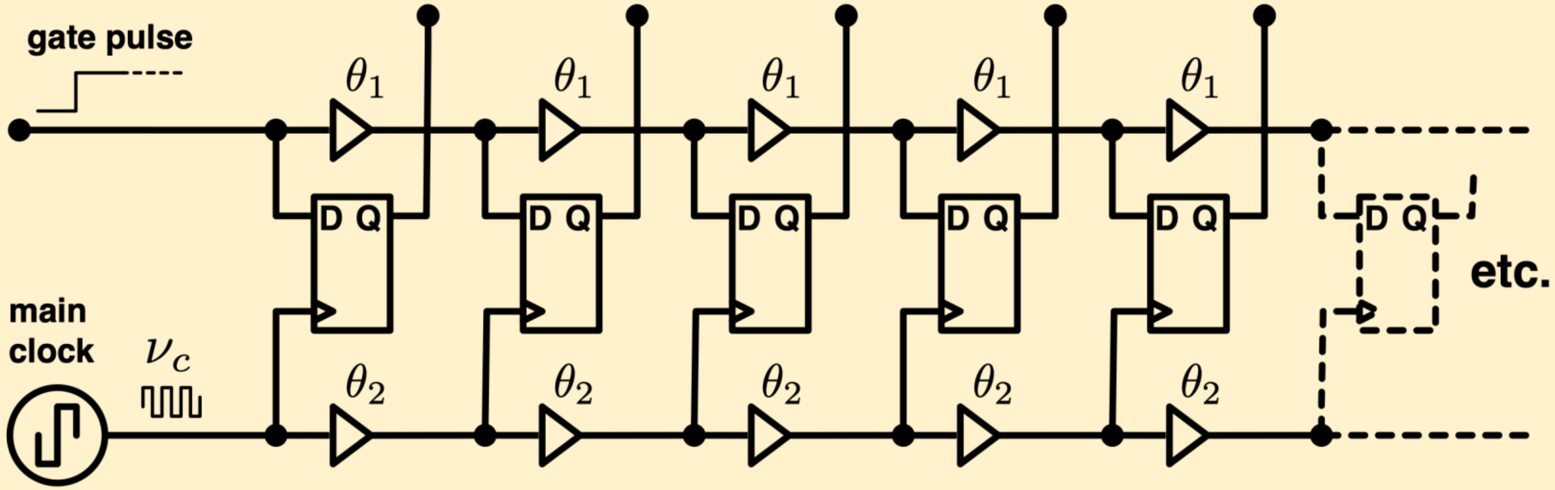
Vernier (enhanced resolution) version

- Delay is on both lines is inevitable
- Just exploit it

$\theta_{eq} = \theta_{ck} - \theta_{in}$

Review article:
J. Kalisz, Metrologia 41 (2004) 17–32

Vernier thermometer-code interpolator



$$\theta_{\text{eq}} = \theta_2 - \theta_1$$

Owing to physical size, both θ_1 and θ_2 are always present

Featured review article: J. Kalisz, Metrologia 41 (2004) 17–32

Ring oscillator

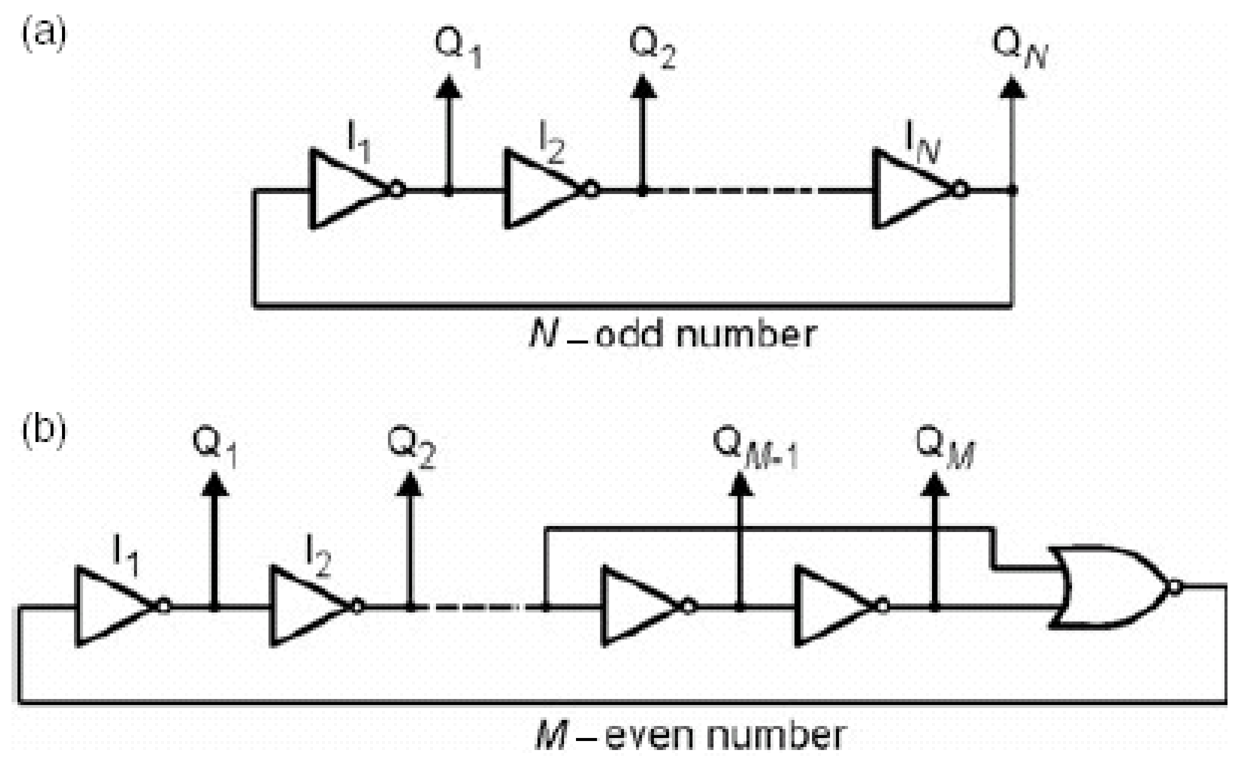
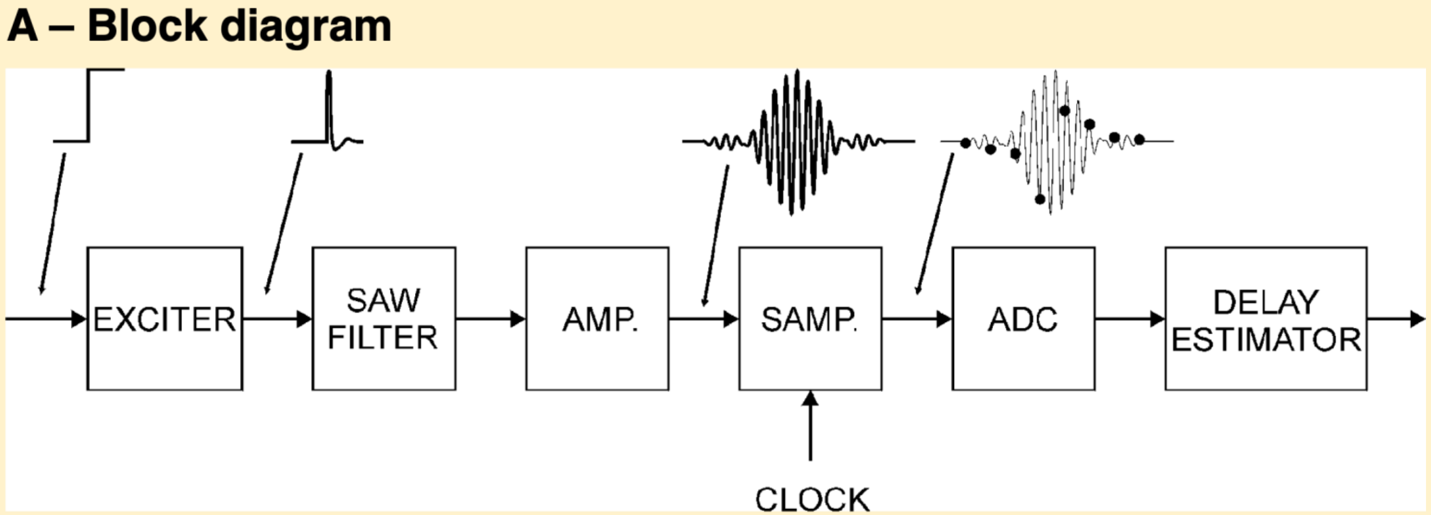


Figure from J. Kalisz, Metrologia 41 (2004) 17–32

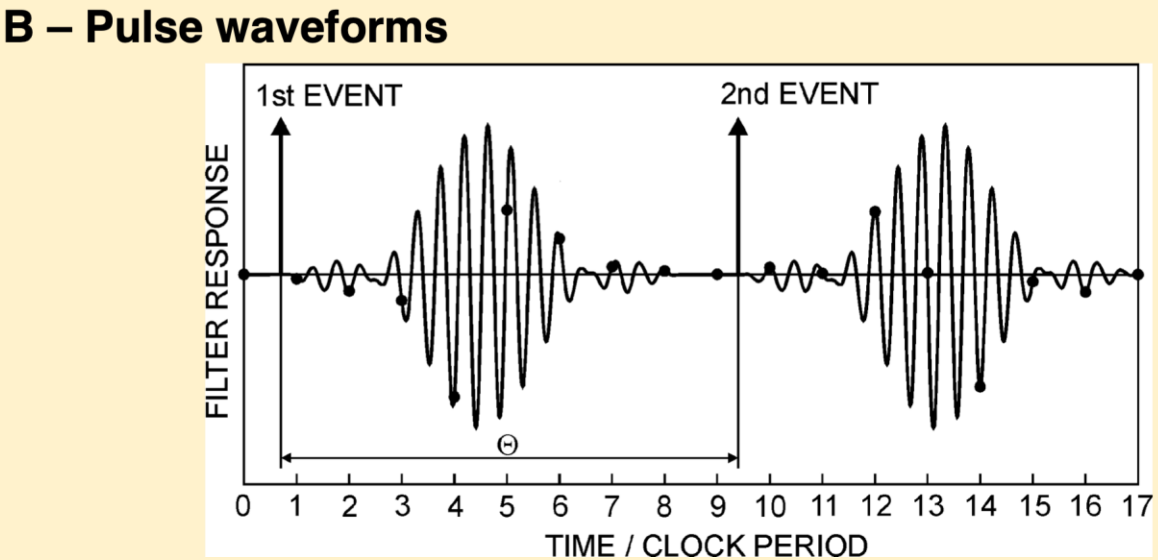
Also used in PLL circuits for clock-frequency multiplication

SAW delay-line interpolator

Panek & Prochazka, Fig.1



Panek & Prochazka, Fig.2

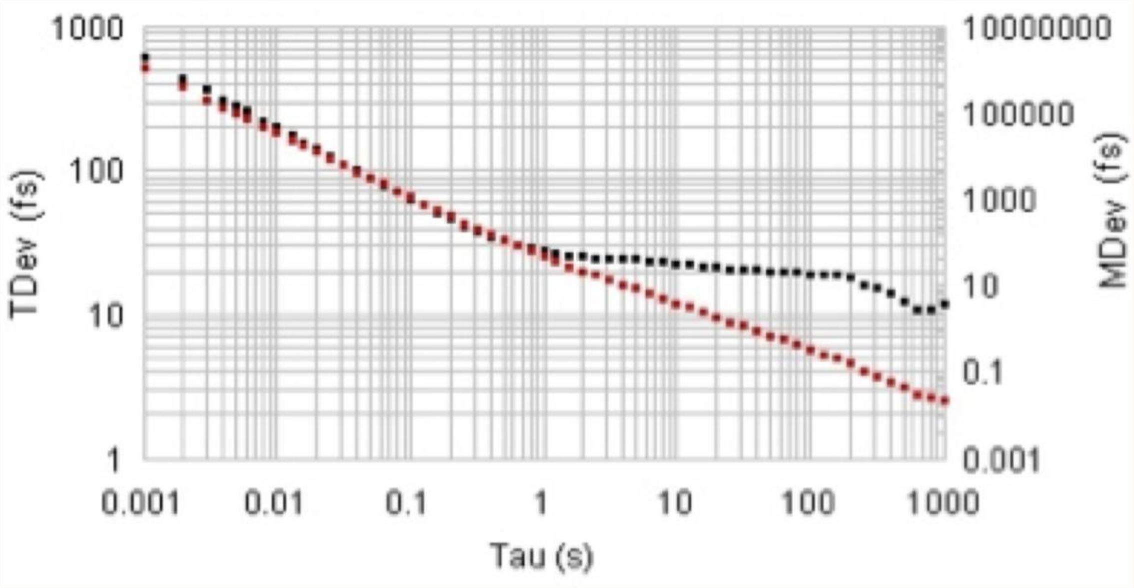


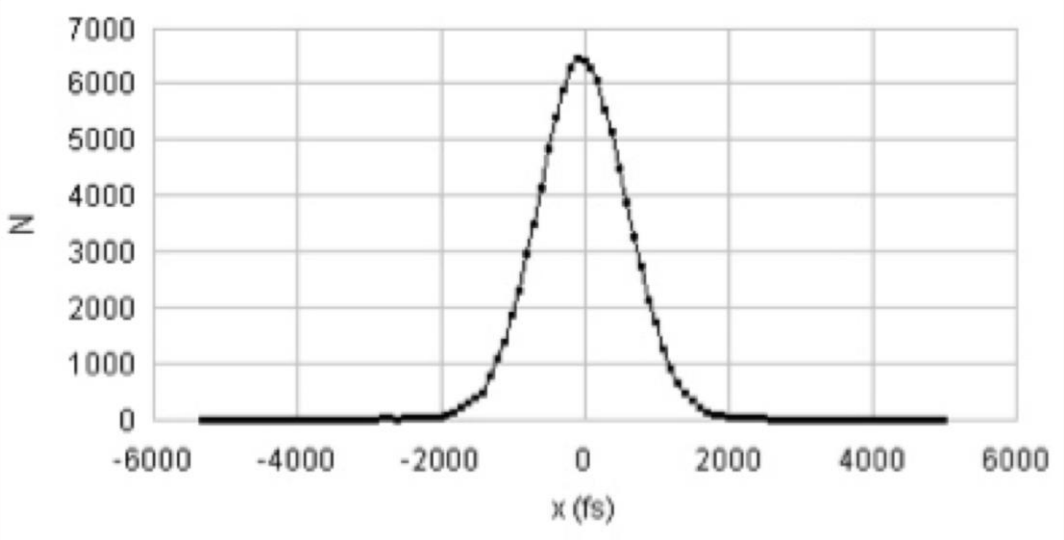
- Dispersion stretches the input pulse
- Sub-sampling and identification of the alias

P. Panek, I. Prochazka, Rev. Sci. Instrum. 78(9) 094701, 2007

Sigma Time STX301

Figures are from the data sheet





- Rumors are that this is none of the methods shown
- No information at all, I'm unable to reverse-engineer

Commercial instruments

Carmel	NK732	3 ps	PCI/PXI time stamp	Ramp
Guide Tech	GT667/668	1 ps	PCI/PXI time stamp	Ramp
Keysight	53230A	20 ps	Lab instrument	Frequency vernier
Lange Electronic	KL-3360	50 ps	Π / Λ, special purpose	Ramp
Lumat			PCI card	Thermometer code
Stanford	SR620	25 ps	Lab instrument	Ramp
Serenum	TDC	6 ps rms	PCB module	FPGA Thermometer code
AMS Group	TDC GPX	22 ps	Chip	
MAXIM	MAX35101	8 ps	Chip	
SPAD Lab	TDC Module		Packaged module	
Texas	THS788	8 ps	Chip	Thermometer code

Lecture 5 ends here